Kraśkiewicz-Pragacz modules and some positivity properties of Schubert polynomials

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Schubert Polynomials

- w: permutation $\rightsquigarrow \mathfrak{S}_w \in \mathbb{Z}[x_1, x_2, \ldots]$
$$\begin{cases} \mathfrak{S}_{ws_i} = \partial_i \mathfrak{S}_w = \frac{\mathfrak{S}_w - s_i \mathfrak{S}_w}{x_i - x_{i+1}} & (\ell(ws_i) < \ell(w)) \\ \mathfrak{S}_{w_0} = x_1^{n-1} x_2^{n-2} \cdots x_{n-1} \end{cases}$$
- $\{\mathfrak{S}_w\} \leftrightarrow$ Schubert classes in the cohomology rings of flag varieties.
- w: grassmannian i.e.
$\exists i,\ w(1) < \cdots < w(i),\ w(i+1) < w(i+2) < \cdots$
$\Longrightarrow\ \mathfrak{S}_w =$ (a Schur polynomial in $x_1, \ldots, x_i$).

Examples for $w \in S_3$:

$$\mathfrak{S}_{123} = 1$$
$$\mathfrak{S}_{132} = x_1 + x_2$$
$$\mathfrak{S}_{213} = x_1$$
$$\mathfrak{S}_{231} = x_1 x_2$$
$$\mathfrak{S}_{312} = x_1^2$$
$$\mathfrak{S}_{321} = x_1^2 x_2$$

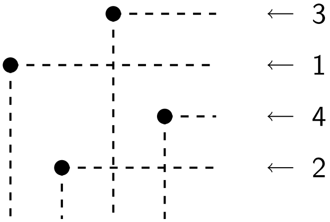
Kraśkiewicz-Pragacz modules

- Schur polynomials: characters of irreducible representations of $\mathfrak{gl}_n(\mathbb{C})$.
- Schubert polynomials: characters of Kraśkiewicz-Pragacz modules.

Kraśkiewicz-Pragacz modules: Definition

- K: field with characteristic 0
- $\mathfrak{b} = \mathfrak{b}_n =$ Lie algebra of $n \times n$ upper triangular matrices
- $K^n = \bigoplus_{1 \le i \le n} K u_i$: vector representation of $\mathfrak{b}$
- $D(w)$: Rothe diagram of a permutation $w \rightsquigarrow \mathcal{S}_w$: KP module

Example: $w = 3142$



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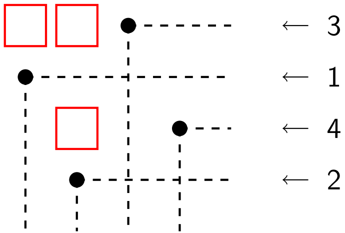

$$D(w) = \{(i, w(j)) : i < j, w(i) > w(j)\}$$
$$= \{(1,1), (1,2), (3,2)\}$$

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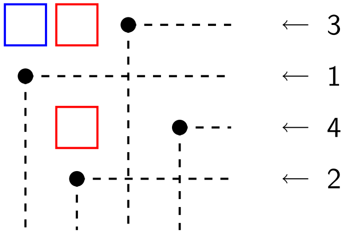


$$u_w = u_1 \otimes (u_1 \wedge u_3) \in K^n \otimes \bigwedge^2 (K^n)$$
$$\mathcal{S}_w = \mathcal{U}(\mathfrak{b}) u_w = \langle u_1 \otimes (u_1 \wedge u_3), u_1 \otimes (u_1 \wedge u_2) \rangle$$

Kraśkiewicz-Pragacz modules: Property

For a $\mathfrak{b}$-module M and $\lambda \in \mathbb{Z}^n$, let
$M_\lambda = \{m \in M : hm = \sum_i \lambda_i h_i \ (\forall h = \mathrm{diag}(h_1, \ldots, h_n) \in \mathfrak{b})\}$ and let
$\mathrm{ch}(M) = \sum_\lambda \dim M_\lambda x^\lambda$: character of M.

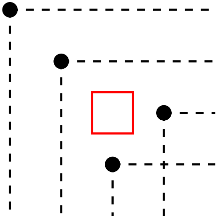
eg. $\mathcal{S}_{3142} = \langle u_1 \otimes (u_1 \wedge u_3), u_1 \otimes (u_1 \wedge u_2) \rangle \rightsquigarrow \mathrm{ch}(\mathcal{S}_{3142}) = x_1^2 x_3 + x_1^2 x_2$.

Theorem (Kraśkiewicz-Pragacz)

$\mathrm{ch}(\mathcal{S}_w) = \mathfrak{S}_w$.

Kraśkiewicz-Pragacz modules: Examples

$w = s_i$: simple transposition

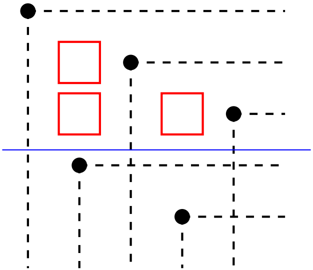


$$u_{s_i} = u_i$$
$$\mathcal{S}_{s_i} = Ku_1 \oplus \cdots \oplus Ku_i$$
$$\mathrm{ch}(\mathcal{S}_{s_i}) = x_1 + \cdots + x_i = \mathfrak{S}_{s_i}$$

Kraśkiewicz-Pragacz modules: Examples

w: grassmannian i.e. $w(1) < \cdots < w(i)$, $w(i+1) < w(i+2) < \cdots$

eg. $w = 13524$



$$u_w = \text{a lowest weight vector in an irreducible representation of } \mathfrak{gl}_i(\mathbb{C})$$
$$\mathcal{S}_w = \text{an irreducible representation of } \mathfrak{gl}_i(\mathbb{C})$$
$$\mathrm{ch}(\mathcal{S}_w) = (\text{a Schur polynomial in } x_1, \ldots, x_i) = \mathfrak{S}_w$$

Remark: the examples we have seen are all special cases of Demazure modules, but in general they are different (equal only for 2143-avoiding w).

Schubert positivity

- For permutations w, v, is the product $\mathfrak{S}_w \mathfrak{S}_v$ a positive sum of Schubert polynomials?
  ($\rightsquigarrow$ Yes: classical, one of very fundamental properties of Schubert polynomials. Previously known proof is through the cohomology ring of flag varieties)

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- For a partition λ and a permutation w, is the plethysm $s_\lambda[\mathfrak{S}_w] = s_\lambda(x^\alpha, x^\beta, \dots)$ ($\mathfrak{S}_w = x^\alpha + x^\beta + \cdots$) a positive sum of Schubert polynomials?
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How are such questions related with KP modules?

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Recall the case of Schur functions: the Schur positivities of the product $s_\lambda s_\mu$ and the plethysm $s_\lambda[s_\mu]$ can be both explained by interpreting them as characters of certain modules over $\mathfrak{gl}_n(\mathbb{C})$.

Schubert positivity and KP modules

- For permutations w, v, is the product $\mathfrak{S}_w \mathfrak{S}_v$ a positive sum of Schubert polynomials?
  $\leftrightsquigarrow$ For permutations w, v, does the tensor product module $\mathcal{S}_w \otimes \mathcal{S}_v$ have a KP filtration, i.e. a filtration $\mathcal{S}_w \otimes \mathcal{S}_v = M_r \supset M_{r-1} \supset \cdots \supset M_0 = 0$ such that each M_i/M_{i-1} is isomorphic to some KP module?
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So,

Question

When does a $\mathfrak{b}$-module M have a KP filtration?

Criterion for KP filtration

For $k \in \mathbb{Z}$ let $K_{\rho+k\mathbf{1}}$ be the 1-dimensional $\mathfrak{b}$-module with weight $\rho + k\mathbf{1} = (n-1+k, n-2+k, \ldots, k) \in \mathbb{Z}^n$.

Theorem (W.)

M has a KP filtration
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Key step: introduce an order $\leq$ on $\mathbb{Z}^n$ using the lexicographic order on the inverse of the permutation, for example,

$$\underset{\substack{\| \\ (1,3,0,1,0)}}{\mathrm{code}(25143)} \leq \underset{\substack{\| \\ (0,3,2,0,0)}}{\mathrm{code}(15423)} \quad \text{since} \quad 25143^{-1} \underset{\mathrm{lex}}{\geq} 15423^{-1}.$$

Proposition (W.)

$\mathcal{S}_w$ is the projective cover of $K_{\mathrm{code}(w)}$ in $\mathcal{C}_{\leq \mathrm{code}(w)}$, the category of all modules whose weights are $\leq \mathrm{code}(w)$.

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Corollary

- $M = M_1 \oplus \cdots \oplus M_r$ has KP filtration iff each M_i does.
- If $0 \to L \to M \to N \to 0$ & M, N have KP filtrations then so does L.

With this corollary:

- Plethysm question can be reduced to Tensor product question since $\mathcal{S}_w^{\otimes m} = \bigoplus_{\lambda \vdash m} \left(s_\lambda(\mathcal{S}_w)\right)^{f^\lambda}$.
- Tensor product question can be reduced to a simpler case $\mathcal{S}_w \otimes \mathcal{S}_{s_i}$ where the expansion of its character into a sum of Schubert polynomials is explicitly known (Monk's formula).

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The multiplicities of Schubert polynomials in these expansions can be also described by KP modules: if M has a KP filtration, then it can be seen that the number of times $\mathcal{S}_w$ ($w \in S_n$) appears in M is given by $\dim Hom(M \otimes S_{w_0 w}, K_\rho)$. So in this case we have,

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- If $s_\lambda[\mathfrak{S}_w] = \sum d^v_{\lambda w}\mathfrak{S}_v$, then

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