Preface

The study of Schubert polynomials is one of the main subjects in algebraic combinatorics. One of the possible methods for studying Schubert polynomials is through the modules introduced by Kraśkiewicz and Pragacz. In [10], for each permutation w they introduced a certain module $\mathcal{S}_w$ over the Lie algebra $\mathfrak{b}$ of all upper triangular matrices whose character is the corresponding Schubert polynomial $\mathfrak{S}_w$. In this paper we call them *Kraśkiewicz-Pragacz modules* or *KP modules* for short.

Schubert polynomials can be regarded as a generalization of Schur functions. Many positivity properties are known for Schur functions. One of the most classical examples is the Schur positivity of the product $s_\lambda s_\mu$ of Schur functions, i.e. the product of Schur functions always expands into a positive sum of Schur functions. Another such example is the positivity of so-called *plethysms* of Schur functions. Plethysm is another kind of product operation (other than the ordinary multiplication) defined on symmetric functions, corresponding to the composition of representations of general linear Lie algebras. It is also known that the plethysm $s_\lambda[s_\mu]$ of Schur functions is always Schur positive.

One of the main motivations for our study of KP modules is the corresponding generalizations of these positivity properties to Schubert polynomials. The positivity of the product of Schubert polynomials is classically known: the product $\mathfrak{S}_w\mathfrak{S}_v$ of Schubert polynomials always expands into a positive sum of Schubert polynomials. The only previously known proof for this positivity is through a geometric interpretation of Schubert polynomials, i.e. through the interpretation of the coefficients c^u_{wv} appearing in $\mathfrak{S}_w\mathfrak{S}_v = \sum_u c^u_{wv}\mathfrak{S}_u$ as the number of intersection points of certain subvarieties in a flag variety. It is a long-standing problem in algebraic combinatorics to give a combinatorial positive rule for these coefficients c^u_{wv}, like Littlewood-Richardson rule in the Schur-function case. One of our results gives a proof for the positivity of the coefficients and an interpretation of these coefficients from yet another point of view, i.e. from a representation theoretic viewpoint using KP modules.

Plethysms can be also generalized to the case of Schubert polynomials: the plethysm $s_\lambda[\mathfrak{S}_w]$ of a Schur function with a Schubert polynomial can be defined in the same way as the plethysms of two Schur functions. Our results also give a proof to the fact that this plethysm always expands into a positive sum of Schubert polynomials, which was not known before.

In studying such Schubert positivity phenomena, it is important to consider the class of $\mathfrak{b}$-modules having *KP filtrations*, i.e. filtrations of $\mathfrak{b}$-modules whose successive quotients are isomorphic to KP modules. Since KP modules have Schubert polynomials as their characters, if a module M has a KP filtration then its character is Schubert positive. So for example if we show that the tensor product $\mathcal{S}_w \otimes \mathcal{S}_v$ of KP modules and the Schur functor image $s_\lambda(\mathcal{S}_w)$ of a KP module have KP filtrations then it gives proofs for the Schubert positivities of the product $\mathfrak{S}_w\mathfrak{S}_v$ and the plethysm $s_\lambda[\mathfrak{S}_w]$ respectively.

In this paper we study the class of modules having KP filtrations using the theory of highest weight categories ([4]). We see that certain categories of $\mathfrak{b}$-modules can be equipped with structures of highest weight categories so that the standard objects are KP modules (Theorem 2.3.1). Note that the order relation on the set of weights is not the usual root order (see the beginning of Section 2.2). Then using the generalities on highest weight categories we show that the tensor product modules $\mathcal{S}_w \otimes \mathcal{S}_v$ and Schur functor images $s_\lambda(\mathcal{S}_w)$ actually have KP filtrations (Theorem 3.1.1); this, as explained above, gives a new proof for the positivity of products as well as a new result concerning the plethysms of Schur functions with Schubert polynomials. Highest weight theory for KP modules enables us to reduce the problems above on tensor products and Schur functor images to simpler problems. For example, the problem on Schur functor images can be easily reduced to the problem on tensor products because $s_\lambda(\mathcal{S}_w)$ is a direct summand of $\mathcal{S}_w^{\otimes|\lambda|}$ and by the generalities of highest weight categories the existence of KP filtrations inherits to direct summands. Also the tensor product problem can be reduced to very simple cases corresponding to Monk's formula for Schubert polynomials using the generalities of highest weight categories. For the details see Section 3.1.

Our works relating KP modules with the notion of highest weight categories were strongly inspired by similar works on Demazure modules ([18], [22], [23, §3]). Demazure modules (for $\mathfrak{gl}_n$) and KP modules seem to have many striking similarities: they are both families of $\mathfrak{b}$-modules parametrized by their lowest weights and they both well fit into the theory of highest weight categories. Also we get a presentations of KP modules (Theorem 2.1.1) which are very similar to the presentations of Demazure modules ([9, Theorem 3.4]) by Joseph (note that KP modules are, despite their similarities with Demazure modules, not special cases of Demazure modules: see Example 1.2.5).

We also show that a special case of the highest weight categories we introduce, namely the one denoted by $\mathcal{C}_n$ in this paper, have particularly nice properties (Theorem 4.1.1, Theorem 4.2.1): its Ringel dual is equivalent to $\mathcal{C}_n$ itself, and the natural autoequivalence on the subcategory $\mathcal{C}_n^\Delta$ of modules having KP filtrations preserves a certain tensor product operation on $\mathcal{C}_n^\Delta$. The correspondence of the standard objects under the Ringel duality is given by $\mathcal{S}_w \mapsto \mathcal{S}_{w_0 w w_0}$ $(w \in S_n)$, which suggests some connection with the involution on the cohomology ring of the flag manifold $H^\bullet(Fl(\mathbb{C}^n)) \cong \mathbb{Z}[x_1, \ldots, x_n]/(e_i(x_1, \ldots, x_n))_{1 \le i \le n}$ given by $x_i \mapsto -x_{n+1-i}$ (see Proposition 1.1.4 and Remark 1.1.5). One of the interesting consequences of this duality is a kind of symmetry relation on the extension groups between KP modules: we have $\mathrm{Ext}^i(\mathcal{S}_w, \mathcal{S}_v) \cong \mathrm{Ext}^i(\mathcal{S}_{w_0 v w_0}, \mathcal{S}_{w_0 w w_0})$ for $w, v \in S_n$ where $w_0 \in S_n$ is the longest element.

This paper is organized as follows. In Section 1 we prepare basic definitions and results on Schubert polynomials, KP modules and highest weight categories. In Section 2 we relate the notion of highest weight categoires with KP modules: we show that certain categories of $\mathfrak{b}$-modules admit highest weight structures so that the standard objects are KP modules. In Section 3 we utilize the highest weight structure developed in the previous section to show that the tensor product modules $\mathcal{S}_w \otimes \mathcal{S}_v$ and Schur functor images $s_\lambda(\mathcal{S}_w)$ actually have KP filtrations. We also give constructions of explicit filtrations for the tensor product modules corresponding to the Pieri and dual Pieri rules for Schubert polynomials ([2], [25]). Note that the Pieri rule for KP modules actually gives another proof for the existence of desired highest weight structure for $\mathfrak{b}$-modules. In Section 4 we focus on a special case $\mathcal{C}_n$ of our highest weight categories where the KP modules under consideration are the ones $\mathcal{S}_w$ with $w \in S_n$. We show that the Ringel dual of $\mathcal{C}_n$ is equivalent to itself, and the natural autoequivalence $F : \mathcal{C}_n^\Delta \to \mathcal{C}_n^\Delta$ given by the Ringel duality is in some sense compatible with tensor product, i.e. $F((M \otimes N)^{\Lambda_n}) \cong (FM \otimes FN)^{\Lambda_n}$ where L^{Λ_n} denotes the largest quotient of L whose weights are in Λ_n.

The structure of the arguments for the first part is slightly modified from the submitted version of the paper in order to separate the general theory of highest weight categories from particular arguments on special properties of KP modules. Also, the proof of Monk's rule for KP modules (Proposition 3.1.2) used in proving the existence of KP filtrations for tensor product modules is modified to use a more general result on KP modules corresponding to the Pieri rule (Section 3.2).

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Contents

1 Preliminaries

1.1 Permutations and Schubert polynomials

In this subsection we review definitions and basic properties of Schubert polynomials. We use [14] as a main reference. For the original source of these properties see the references in [12], [14] and [17].

Let $\mathbb{Z}_{>0}$ and $\mathbb{Z}_{\geq 0}$ denote the set of all positive and nonnegative integers respectively. By a *permutation* w we mean a bijection from $\mathbb{Z}_{>0}$ to itself which fixes all but finitely many points. Let S_∞ denote the group of all permutations. The *graph* of a permutation w is the set $\{(i, w(i)) : i \in \mathbb{Z}_{>0}\} \subset \mathbb{Z}_{>0}^2$. For a positive integer n let $S_n = \{w \in S_\infty, w(i) = i \ (i > n)\}$ and $S_\infty^{(n)} = \{w \in S_\infty, w(n+1) < w(n+2) < \cdots\}$.

We sometimes write a permutation in its one-line form: i.e., we write $[w(1)\, w(2)\, \cdots]$ to express $w \in S_\infty$. If $w \in S_n$, we may write $[w(1)\, w(2)\, \cdots\, w(n)]$ to mean w.

For $i < j$, let t_{ij} denote the permutation which exchanges i and j and fixes all other points. Let $s_i = t_{i,i+1}$. For a permutation w, let $\ell(w) = \#\{i < j : w(i) > w(j)\}$ and $\mathrm{sgn}(w) = (-1)^{\ell(w)}$. For a permutation w and $p < q$, if $\ell(wt_{pq}) = \ell(w) + 1$ we write $wt_{pq} \gtrdot w$. It is well known that this condition is equivalent to saying that $w(p) < w(q)$ and there exists no $p < r < q$ satisfying $w(p) < w(r) < w(q)$.

For $w \in S_\infty^{(n)}$ we define $\mathrm{code}(w) = (\mathrm{code}(w)_1, \ldots, \mathrm{code}(w)_n) \in \mathbb{Z}_{\geq 0}^n$ by $\mathrm{code}(w)_i = \#\{j : i < j, w(i) > w(j)\}$: this is usually called the *Lehmer code* and it uniquely determines w. Note that if $w \in S_n$ we have $\mathrm{code}(w) \in \Lambda_n := \{(a_1, \ldots, a_n) \in \mathbb{Z}^n : 0 \leq a_i \leq n - i\}$. For $\lambda \in \mathbb{Z}_{\geq 0}^n$ we define $\mathrm{perm}(\lambda) \in S_\infty^{(n)}$ as the permutation satisfying $\mathrm{code}(\mathrm{perm}(\lambda)) = \lambda$.

For a permutation w we assign its *inversion diagram* defined by $I(w) = \{(i, j) : i < j, w(i) > w(j)\}$. Note that if $w \in S_\infty^{(n)}$ then $I(w) \subset \{1, \ldots, n\} \times \mathbb{Z}_{>0}$.

For a polynomial $f = f(x_1, x_2, \ldots)$ and $i \in \mathbb{Z}_{>0}$ define $\partial_i f = \frac{f - s_i f}{x_i - x_{i+1}}$. For a permutation w we can assign its *Schubert polynomial* $\mathfrak{S}_w \in \mathbb{Z}[x_1, x_2, \ldots]$ which is recursively defined by

- $\mathfrak{S}_w = x_1^{n-1} x_2^{n-2} \cdots x_{n-1}$ if $w(1) = n, w(2) = n-1, \ldots, w(n) = 1$ and $w(i) = i \ (i > n)$ for some n, and
- $\mathfrak{S}_{ws_i} = \partial_i \mathfrak{S}_w$ if $\ell(ws_i) < \ell(w)$.

It is known ([14, (4.11)]) that $\{\mathfrak{S}_w : w \in S_\infty^{(n)}\}$ (resp. $\{\mathfrak{S}_w : w \in S_n\}$) constitutes a $\mathbb{Z}$-linear basis for $\mathbb{Z}[x_1, \ldots, x_n]$ (resp. $\bigoplus_{\lambda \in \Lambda_n} \mathbb{Z} x_1^{\lambda_1} \cdots x_n^{\lambda_n}$). It is also known ([17, Proposition 2.5.3, Corollary 2.5.6]) that the latter constitutes a $\mathbb{Z}$-linear basis for the quotient ring $H_n = \mathbb{Z}[x_1, \ldots, x_n]/I_n$, where $I_n \subset \mathbb{Z}[x_1, \ldots, x_n]$ is the ideal generated by all symmetric polynomials in $x_1, \ldots, x_n$ without constant terms.

Below we list some properties of Schubert polynomials used in this paper. The following identity is known as *Monk's formula*:

Proposition 1.1.1 ([14, (4.15″)]). *Let $w \in S_\infty$ and $\nu \in \mathbb{Z}_{>0}$. Then $\mathfrak{S}_w \mathfrak{S}_{s_\nu} = \sum \mathfrak{S}_{wt_{pq}}$ where the sum is over all (p, q) such that $p \leq \nu < q$ and $wt_{pq} \gtrdot w$.*

Generalizations of Monk's formula include the Pieri and dual Pieri rules for Schubert polynomials which give expansions of products of Schubert polynomials with complete symmetric functions $h_d(x_1, \ldots, x_i)$ and elementary symmetric functions $e_d(x_1,\dots,x_n)$ respectively. We will present these rules later in this paper.

One of the consequences of Monk's formula is the following recursion for Schubert polynomials known as *transition*:

Proposition 1.1.2 ([14, (4.16)]). *Let $w \in S_\infty \smallsetminus \{\mathrm{id}\}$. Let $j \in \mathbb{Z}_{>0}$ be the maximal integer such that $w(j) > w(j+1)$ and take $k > j$ maximal with $w(j) > w(k)$. Let $v = wt_{jk}$. Let $i_1 < \cdots < i_A$ be the all integers less than j such that $vt_{i_a j} \gtrdot v$, and let $w^{(a)} = vt_{i_a j}$. Then*

$$\mathfrak{S}_w = x_j \mathfrak{S}_v + \sum_{a=1}^{A} \mathfrak{S}_{w^{(a)}}.$$

Note that if $w \in S_\infty^{(n)}$ then v and $w^{(1)},\dots,w^{(A)}$ in the proposition above are also in $S_\infty^{(n)}$.

Schubert polynomials also satisfy the following *Cauchy identity*:

Proposition 1.1.3 ([14, (5.10)]). $\sum_{w\in S_n} \mathfrak{S}_w(x)\mathfrak{S}_{ww_0}(y) = \prod_{i+j\le n}(x_i+y_j)$ *where $w_0 = [n\ n-1\ \cdots\ 1] \in S_n$.*

We also need the following basic facts:

Proposition 1.1.4. *Let $\iota : H_n \to H_n$ be the ring automorphism given by $\overline{x_i} \mapsto -\overline{x_{n+1-i}}$ where $\overline{x_i} = x_i \bmod I_n$. Then for $w \in S_n$, $\iota(\mathfrak{S}_w) = \mathfrak{S}_{w_0ww_0}$.*

Remark 1.1.5. The automorphism ι corresponds to the map between flag varieties which takes a flag to its dual flag: see eg. [7, §10.6, Exercise 13]

Proof. First note that $\iota\circ\partial_i\circ\iota = \partial_{n-i}$. Thus we only have to check the proposition for $w = w_0$.

Since the only elements in $H_n = \bigoplus_{w\in S_n}\mathbb{Z}\mathfrak{S}_w$ with degree $\binom{n}{2}$ are the constant multiples of $\mathfrak{S}_{w_0}$, we see that $\iota(\mathfrak{S}_{w_0})$ is a constant multiple of $\mathfrak{S}_{w_0}$. Let $(i_1,\dots,i_l)$ be a longest word, i.e. $l = \ell(w_0)$ and $w = s_{i_1}\cdots s_{i_l}$. Note that $(n-i_1,\dots,n-i_l)$ is also a longest word. We have $\partial_{i_1}\cdots\partial_{i_l}\mathfrak{S}_{w_0} = \mathfrak{S}_{\mathrm{id}} = 1$ and $\partial_{i_1}\cdots\partial_{i_l}\iota(\mathfrak{S}_{w_0}) = (\iota\partial_{n-i_1}\iota)\cdots(\iota\partial_{n-i_l}\iota)\iota(\mathfrak{S}_{w_0}) = \iota(\partial_{n-i_1}\cdots\partial_{n-i_l}\mathfrak{S}_{w_0}) = 1$. Thus $\iota(\mathfrak{S}_{w_0}) = \mathfrak{S}_{w_0}$. □

Proposition 1.1.6. *For $w \in S_\infty^{(n)} \smallsetminus S_n$ we have $\mathfrak{S}_w \in I_n$.*

Proof. Since $\partial_i I_n \subset I_n$ for any $1 \le i \le n-1$, it suffices to show that the proposition holds in the case $w(1) > \cdots > w(n)$. Since in this case $\mathfrak{S}_w = x_1^{w(1)-1}x_2^{w(2)-1}\cdots x_n^{w(n)-1}$ it is enough to show $x_1^n \in I_n$. This is immediate from the equation $\prod_{i=2}^n(1-\overline{x_i}u) = \frac{1}{1-\overline{x_1}u} = \sum_{j\ge0}\overline{x_1}^j u^j$ in $H_n[[u]]$ since the LHS has no terms of degrees $\ge n$ in u. □

1.2 Kraśkiewicz-Pragacz modules

Let K be a field of characteristic zero. Let $\mathfrak{b} = \mathfrak{b}_n$ be the Lie algebra of all $n\times n$ upper triangular K-matrices. and let $\mathfrak{h} \subset \mathfrak{b}$ and $\mathfrak{n}^+ \subset \mathfrak{b}$ be the subalgebra of all diagonal matrices and the subalgebra of all strictly upper triangular matrices respectively. For a Lie algebra $\mathfrak{g}$ let $\mathcal{U}(\mathfrak{g})$ denote its universal enveloping algebra.

For a $\mathfrak{b}$-module M and $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{Z}^n$, let $M_\lambda = \{m \in M : hm = \langle\lambda, h\rangle m \ (\forall h \in \mathfrak{h})\}$ where $\langle\lambda, h\rangle = \sum \lambda_i h_i$. M_λ is called the λ-*weight space* of M. If $M_\lambda \neq 0$ then λ is said to be a *weight* of M. If $M = \bigoplus_{\lambda\in\mathbb{Z}^n} M_\lambda$ and each M_λ has finite dimension, then we call that M is a *weight* $\mathfrak{b}$*-module* and define $\mathrm{ch}(M) = \sum_\lambda \dim M_\lambda x^\lambda$ ($x^\lambda = x_1^{\lambda_1} \cdots x_n^{\lambda_n}$). From here we only consider weight $\mathfrak{b}$-modules, and for weight $\mathfrak{b}$-modules M and N, $\mathrm{Ext}^i(M, N)$ always mean the Ext groups taken in the category of all weight $\mathfrak{b}$-modules (not the whole $\mathfrak{b}$-modules).

For $1 \leq i \leq j \leq n$, let $e_{ij} \in \mathfrak{b}$ be the matrix with 1 at the (i, j)-position and all other coordinates 0. Let $\rho = (n-1, n-2, \ldots, 0) \in \mathbb{Z}^n$ and $\mathbf{1} = (1, \ldots, 1) \in \mathbb{Z}^n$. Also let $\epsilon_i = (0, 0, \ldots, 1, \ldots, 0, 0) \in \mathbb{Z}^n$ with 1 at the i-th position, and let $\alpha_{ij} = \epsilon_i - \epsilon_j$ for $1 \leq i < j \leq n$. Note that if M is a $\mathfrak{b}$-module and $x \in M_\lambda$ then $e_{ij}x \in M_{\lambda+\alpha_{ij}}$.

For $\lambda \in \mathbb{Z}^n$, let K_λ denote the one-dimensional $\mathfrak{b}$-module where $h \in \mathfrak{h}$ acts by $\langle\lambda, h\rangle$ and $\mathfrak{n}^+$ acts by 0. Note that every finite-dimensional weight $\mathfrak{b}$-modules admits a filtration by these one-dimensional modules.

In [10] Kraśkiewicz and Pragacz defined certain $\mathfrak{b}$-modules which we call here *Kraśkiewicz-Pragacz modules* or *KP modules*. Here we use the following definition.

Let $w \in S_\infty^{(n)}$. Let $K^n = \bigoplus_{1\leq i\leq n} Ku_i$ be the vector representation of $\mathfrak{b}$: i.e. $e_{ij}u_k = \delta_{jk}u_i$. For each $j \in \mathbb{Z}_{>0}$, let $l_j = l_j(w) = \#\{i : (i, j) \in I(w)\}$, $\{i : (i, j) \in I(w)\} = \{i_1, \ldots, i_{l_j}\}$ ($i_1 < \cdots < i_{l_j}$), and $u_w^{(j)} = u_{i_1} \wedge \cdots \wedge u_{i_{l_j}} \in \bigwedge^{l_j} K^n$. Note that $u_w^{(j)}$ is actually in $\bigwedge^{l_j} K^{\min\{n,j-1\}}$ where $K^i = Ku_1 \oplus \cdots \oplus Ku_i \subset K^n$. Let $u_w = u_w^{(1)} \otimes u_w^{(2)} \otimes \cdots \in \bigwedge^{l_1} K^n \otimes \bigwedge^{l_2} K^n \otimes \cdots$. Then the KP module $\mathcal{S}_w$ associated to w is defined as $\mathcal{S}_w = \mathcal{U}(\mathfrak{b})u_w \subset \bigwedge^{l_1} K^n \otimes \bigwedge^{l_2} K^n \otimes \cdots$.

Remark 1.2.1. It is also possible to define KP modules similarly using the so-called *Rothe diagram* $D(w) = \{(i, w(j)) : i < j, w(i) > w(j)\}$ of w instead of $I(w)$. Since $I(w)$ and $D(w)$ differ only by a rearrangement of columns it does not matter which to use. $D(w)$ has an advantage that it is easier to visualize for concrete examples: if one draw rays downward and to the right from the position $(i, w(i))$ ($i = 1, 2, \ldots$), then the remaining boxes give $D(w)$ (see the figure below). Also, in [6] a basis for $\mathcal{S}_w$ is constructed using certain labellings of the Rothe diagram.

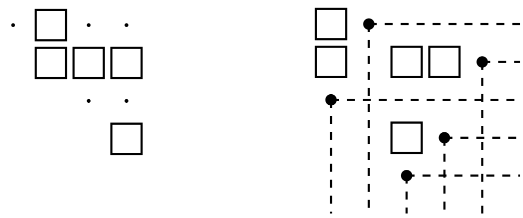


Figure 1: the inversion diagram (left) and the Rothe diagram (right) of the same permutation [25143].

As proved by Kraśkiewicz and Pragacz in [11], KP modules have the following property:

Theorem 1.2.2 ([11, Remark 1.6 and Theorem 4.1])**.** *$\mathcal{S}_w$ is a weight module and $\mathrm{ch}(\mathcal{S}_w) = \mathfrak{S}_w$ (so in particular, $\dim \mathcal{S}_w = \mathfrak{S}_w(1) := \mathfrak{S}_w(1, 1, \ldots, 1)$).*

Example 1.2.3. If $w = s_i$, then $I(s_i) = \{(i, i+1)\}$, $u_{s_i} = u_i$ and $\mathcal{S}_{s_i} = \bigoplus_{1 \le j \le i} K u_j = K^i$. So $\mathrm{ch}(\mathcal{S}_{s_i}) = x_1 + \cdots + x_i = \mathfrak{S}_{s_i}$.

Example 1.2.4. More generally, if w is grassmannian, i.e. there exists a k such that $w(1) < \cdots < w(k)$ and $w(k+1) < w(k+2) < \cdots$, then the inversion diagram $I(w)$ of w is a "French-notation" Young diagram (see figure below). Thus in this case, u_w is a lowest-weight vector in a certain irreducible representation of $\mathfrak{gl}_k$, and $\mathcal{S}_w$ is equal to this representation (seen as a representation of $\mathfrak{b}_n$ through the morphism $\mathfrak{b}_n \ni e_{pq} \mapsto \begin{cases} e_{pq} & (q \le k) \\ 0 & (q > k) \end{cases} \in \mathfrak{gl}_k$). This reflects the fact that the Schubert polynomial indexed by a grassmannian permutation is a Schur polynomial.

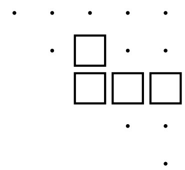


Figure 2: the inversion diagram of a grassmannian permutation [136245] is a French-style Young diagram of shape $(3, 1)$.

Example 1.2.5. More generally, if w is 2143-avoiding, then it can be seen (using the fact ([14, (1.27)]) that the rows of $I(w)$ for 2143-avoiding w is totally preordered by inclusion) that u_w is an extremal vector in an irreducible representation of $\mathfrak{gl}_n$. Thus in this case the corresponding KP module $\mathcal{S}_w$ is isomorphic to a Demazure module of $\mathfrak{b}$: i.e. a module generated by an extremal vector of an irreducible representation of $\mathfrak{gl}_n$. Note that this corresponds to the result first obtained by Lascoux and Schutzenberger ([13, Theorem 5], [12, Corollary 10.5.2]) that Schubert polynomials with 2143-avoiding indices are equal to certain key polynomials.

On the other hand, consider $w = [2143]$. Then $I(w) = \{(1,2), (3,4)\}$, $u_w = u_1 \otimes u_3$, $\mathcal{S}_w = \bigoplus_{1 \le i \le 3} K(u_1 \otimes u_i) = K^1 \otimes K^3$ and $\mathrm{ch}(\mathcal{S}_w) = x_1(x_1 + x_2 + x_3) = \mathfrak{S}_w$. Note that in this case $\mathcal{S}_w$ is *not* isomorphic to the Demazure module with the same lowest weight: $\mathcal{S}_w$ is three-dimensional while the Demazure module with the same lowest weight is two-dimensional. [1] In general, $\mathcal{S}_w$ is isomorphic to the Demazure module $\mathrm{Demaz}(\mathrm{code}(w))$ with lowest weight $\mathrm{code}(w)$ if and only if w is 2143-avoiding. We also note here that there always exists a surjection from $\mathcal{S}_w$ to $\mathrm{Demaz}(\mathrm{code}(w))$: this can be seen using the result from the next section and [9, Theorem 3.4].

In this paper we have to slightly extend the notion of Schubert polynomials and KP modules. For $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{Z}^n$, we define the Schubert polynomial and the KP module associated to λ as follows. For $\lambda \in \mathbb{Z}^n_{\ge 0}$, let $\mathfrak{S}_\lambda = \mathfrak{S}_w$ and $\mathcal{S}_\lambda = \mathcal{S}_w$ where $w = \mathrm{perm}(\lambda)$. For a general $\lambda \in \mathbb{Z}^n$, take $k \in \mathbb{Z}$ so that $\lambda + k\mathbf{1} \in \mathbb{Z}^n_{\ge 0}$, and we define $\mathfrak{S}_\lambda = x^{-k\mathbf{1}} \mathfrak{S}_{\lambda + k\mathbf{1}}$ and $\mathcal{S}_\lambda = K_{-k\mathbf{1}} \otimes \mathcal{S}_{\lambda + k\mathbf{1}}$. Note that this definition does not depend on the choice of k, since if $w \in S^{(n)}_\infty$ and

[1] The KP module $\mathcal{S}_{[2143]}$ in this example is, if not seen as a $\mathcal{U}(\mathfrak{b})$-module but as a $\mathcal{U}(\mathfrak{n}^+)$-module, isomorphic to a Demazure module (say $\mathrm{Demaz}(0,0,1)$); thus the results such as Theorem 2.1.1 for such kind of KP modules follow from known results on Demazure modules. But in fact there also exist KP modules which are, even as $\mathcal{U}(\mathfrak{n}^+)$-modules, not isomorphic to any Demazure modules. An example is $\mathcal{S}_{[13254]} \cong K^2 \otimes K^4$.

$\mathrm{code}(w) = \kappa$, then $\mathrm{perm}(\kappa+\mathbf{1}) = \tilde{w} = [w(1)+1\ \cdots\ w(n)+1\ 1\ w(n+1)+1\ \cdots]$, and $\mathfrak{S}_{\tilde{w}} = x^{\mathbf{1}}\mathfrak{S}_w$ and $\mathcal{S}_{\tilde{w}} = K_{\mathbf{1}} \otimes \mathcal{S}_w$ hold for them. It then follows from the theorem above that $\mathcal{S}_\lambda$ is a weight module and $\mathrm{ch}(\mathcal{S}_\lambda) = \mathfrak{S}_\lambda$ for all $\lambda \in \mathbb{Z}^n$. Note that, since $\mathcal{S}_\lambda$ is generated by an element of weight λ, if $(\mathcal{S}_\lambda)_\mu \neq 0$ (i.e. if x^μ appears in $\mathfrak{S}_\lambda$ with nonzero coefficient) then $\mu \trianglerighteq \lambda$, where $\trianglerighteq$ denote the dominance order: $\mu \trianglerighteq \lambda$ iff $\mu - \lambda = \sum_{i=1}^{n-1} a_i(\epsilon_i - \epsilon_{i+1})$ for some $a_1, \ldots, a_{n-1} \in \mathbb{Z}_{\geq 0}$. We also note here that for any $\mu, \nu \in \mathbb{Z}^n$, the number of $\lambda \in \mathbb{Z}^n$ with $\mu \trianglerighteq \lambda \trianglerighteq \nu$ is finite.

A *KP filtration* of a weight $\mathfrak{b}$-module M is a sequence $0 = M_0 \subset \cdots \subset M_r = M$ of weight $\mathfrak{b}$-modules such that each M_i/M_{i-1} is isomorphic to some KP module $\mathcal{S}_{\lambda^{(i)}}$ ($\lambda^{(i)} \in \mathbb{Z}^n$). Note that if M has a KP filtration then $\mathrm{ch}(M)$ is a positive sum of Schubert polynomials $\mathfrak{S}_\lambda$ ($\lambda \in \mathbb{Z}^n$).

1.3 Highest weight categories

In this subsection we prepare definitions and some basic facts about highest weight categories. Some of them appear in references such as [4], [20] and [5, Appendix], but we also give proofs for them to adapt to our settings and to ensure self-containedness since the formulations of highest weight categories and their properties vary with references.

Our proof for the criterion for the existence of standard filtrations is along the way in [23, Theorem 3.2.7]. Our treatment of tilting objects and Ringel duality mostly follows [5, Appendix], with some minor changes and improvements on the arguments.

Definition 1.3.1. Let $\mathcal{C}$ be an abelian K-category with enough projectives and injectives, such that every object has finite length. Let $\Lambda = (\Lambda, \leq)$ be a finite poset indexing the simple objects $\{L(\lambda) : \lambda \in \Lambda\}$ in $\mathcal{C}$ (called the *weight poset*). Moreover, assume that a family of objects $\{\Delta(\lambda) : \lambda \in \Lambda\}$ called *standard objects* is given. Then $\mathcal{C} = (\mathcal{C}, \Lambda, \{\Delta(\lambda)\})$ is called *a highest weight category* if the following axioms hold:

(1) $\mathrm{Hom}_{\mathcal{C}}(\Delta(\lambda), \Delta(\mu)) = 0$ unless $\lambda \leq \mu$.

(2) $\mathrm{End}_{\mathcal{C}}(\Delta(\lambda)) \cong K$.

(3) Let $P(\lambda)$ denote the projective cover of $L(\lambda)$. Then there exists a surjection $P(\lambda) \twoheadrightarrow \Delta(\lambda)$ such that its kernel admits a filtration whose successive quotients are of the form $\Delta(\nu)$ ($\nu > \lambda$).

In the following let $\mathcal{C}$ be a highest weight category, Λ be its weight poset, and $L(\lambda), P(\lambda), Q(\lambda)$ and $\Delta(\lambda)$ stand for the simple, projective, injective and standard objects respectively. Also, let $\nabla(\lambda)$ denote the costandard objects, i.e. $\nabla(\lambda)$ is the injective hull of $L(\lambda)$ in $\mathcal{C}_{\leq\lambda}$. We denote the head and socle of an object $M \in \mathcal{C}$ by $\mathrm{hd}\, M$ and $\mathrm{soc}\, M$ respectively.

For an order ideal $\Lambda' \subset \Lambda$ we denote by $\mathcal{C}_{\Lambda'}$ the full subcategory of $\mathcal{C}$ consisting of the objects such that its simple constituents are $L(\lambda)$ ($\lambda \in \Lambda'$). We denote $\mathcal{C}_{\leq\lambda}$ etc. to mean $\mathcal{C}_{\{\mu:\mu\leq\lambda\}}$ etc. For $M \in \mathcal{C}$ let $M^{\Lambda'}$ be the largest quotient of M which is in $\mathcal{C}_{\Lambda'}$, and write $M^{\leq\lambda}$ etc. to mean $M^{\{\mu:\mu\leq\lambda\}}$ etc.

Remark 1.3.2. $\mathrm{Ker}(M \twoheadrightarrow M^{\Lambda'})$ does not have any $L(\lambda)$ $(\lambda \in \Lambda')$ as its quotient: if $\mathrm{Ker}(M \twoheadrightarrow M^{\Lambda'})/N \cong L(\lambda)$ is such a quotient, then M/N would be a quotient of M, its simple constituents are $L(\nu)$ $(\nu \in \Lambda')$, and it is strictly larger than $M^{\Lambda'}$: these contradict to the definition of $M^{\Lambda'}$.

For an $M \in \mathcal{C}$ and $\lambda \in \Lambda$ let $(M : L(\lambda))$ denote the number of times $L(\lambda)$ appears in the simple constituents of M. It can be easily seen that $\dim \mathrm{Hom}(P(\lambda), M) = (M : L(\lambda)) \dim \mathrm{Hom}(P(\lambda), L(\lambda))$.

1.3.1 Basic Facts

Lemma 1.3.3. *There is a surjection $\Delta(\lambda) \twoheadrightarrow L(\lambda)$ such that the simple constituents of the kernel are of the form $L(\mu)$ $(\mu < \lambda)$.*

Proof. First we show that $(\Delta(\lambda) : L(\mu)) \neq 0$ implies $\mu \leq \lambda$. Assume $(\Delta(\lambda) : L(\mu)) \neq 0$. This means $\mathrm{Hom}(P(\mu), \Delta(\lambda)) \neq 0$. Since $P(\mu)$ has a filtration by $\Delta(\nu)$ $(\nu \geq \mu)$ it follows that $\mathrm{Hom}(\Delta(\nu), \Delta(\lambda)) \neq 0$ for some $\nu \geq \mu$. Thus $\mu \leq \nu \leq \lambda$.

Next we see $(\Delta(\lambda) : L(\lambda)) = 1$. Since $\mathrm{Ker}(P(\lambda) \twoheadrightarrow \Delta(\lambda))$ has a filtration by $\Delta(\nu)$ $(\nu > \lambda)$ we see that $\mathrm{Hom}(\mathrm{Ker}(P(\lambda) \twoheadrightarrow \Delta(\lambda)), \Delta(\lambda)) = 0$. Thus we have an exact sequence $0 \to \mathrm{Hom}(\Delta(\lambda), \Delta(\lambda)) \to \mathrm{Hom}(P(\lambda), \Delta(\lambda)) \to \mathrm{Hom}(\mathrm{Ker}(P(\lambda) \twoheadrightarrow \Delta(\lambda)), \Delta(\lambda)) = 0$ and thus $\mathrm{Hom}(P(\lambda), \Delta(\lambda)) \cong \mathrm{End}(\Delta(\lambda)) \cong K$. Therefore $(\Delta(\lambda) : L(\lambda)) = 1$.

Finally we show that $L(\lambda)$ is a quotient of $\Delta(\lambda)$. Since $(\Delta(\lambda) : L(\lambda)) = 1$, there exists an $N \subset \Delta(\lambda)$ and a surjection $f : N \twoheadrightarrow L(\lambda)$. By the projectivity of $P(\lambda)$, the surjection $\pi : P(\lambda) \twoheadrightarrow L(\lambda)$ factors as $\pi = fg$ for some $g : P(\lambda) \to N$. The composition $P(\lambda) \to N \hookrightarrow \Delta(\lambda)$ is nonzero and thus must be a nonzero multiple of the surjection $P(\lambda) \twoheadrightarrow \Delta(\lambda)$ since as we saw above $\mathrm{Hom}(P(\lambda), \Delta(\lambda)) \cong K$. But the image of the composition map above is N, so we get $N = \Delta(\lambda)$. Thus the claim follows. ☐

By the lemma above $\Delta(\lambda) \in \mathcal{C}_{\Lambda'}$ for any order ideal Λ' containing λ. Also from the proof we see $\mathrm{Hom}(P(\lambda), L(\lambda)) \cong K$.

Lemma 1.3.4. $\mathrm{Hom}(\Delta(\lambda), L(\mu)) = 0$ *for* $\mu \neq \lambda$ *and* $\mathrm{Hom}(\Delta(\lambda), L(\lambda)) \cong K$. *Thus in particular* $\mathrm{hd}\, \Delta(\lambda) \cong L(\lambda)$.

Proof. This can be easily seen from the exact sequence $0 \to \mathrm{Hom}(\Delta(\lambda), L(\mu)) \to \mathrm{Hom}(P(\lambda), L(\mu))$ since the last term is 0 for $\mu \neq \lambda$ and K for $\mu = \lambda$. ☐

1.3.2 Projectivities of Standard objects

Lemma 1.3.5. $\mathrm{Ext}^1(\Delta(\lambda), L(\mu)) \neq 0$ *implies $\lambda < \mu$. So $\Delta(\lambda)$ is projective in $\mathcal{C}_{\Lambda'}$ for any order ideal Λ' which contains λ as a maximal element.*

Because the simple constituents of $\Delta(\mu)$ are $L(\nu)$ $(\nu \leq \mu)$ we get as a corollary:

Corollary 1.3.6. $\mathrm{Ext}^1(\Delta(\lambda), \Delta(\mu)) \neq 0$ *implies $\lambda < \mu$.* ☐

Proof of the Lemma 1.3.5. Assume $\mathrm{Ext}^1(\Delta(\lambda), L(\mu)) \neq 0$. Let $M = \mathrm{Ker}(P(\lambda) \twoheadrightarrow \Delta(\lambda))$, so M has a filtration by $\Delta(\nu)$ $(\nu > \lambda)$. By the exact sequence $\mathrm{Hom}(M, L(\mu)) \to \mathrm{Ext}^1(\Delta(\lambda), L(\mu)) \to \mathrm{Ext}^1(P(\lambda), L(\mu)) = 0$ we see that $\mathrm{Hom}(M, L(\mu)) \neq 0$. This implies that $\mathrm{Hom}(\Delta(\nu), L(\mu)) \neq 0$ for some $\nu > \lambda$. So $\mu = \nu > \lambda$ by Lemma 1.3.4. ☐

Since $\operatorname{hd}\Delta(\lambda) \cong L(\lambda)$ by Lemma 1.3.4 we get:

Proposition 1.3.7. *Let $\Lambda' \subset \Lambda$ be a finite order ideal and let $\lambda \in \Lambda'$ be a maximal element. Then $\Delta(\lambda) \twoheadrightarrow L(\lambda)$ is a projective cover in $\mathcal{C}_{\Lambda'}$ (so $\Delta(\lambda) \cong P(\lambda)^{\Lambda'}$).* □

1.3.3 Hom and Ext between Standard and Costandard Objects

Proposition 1.3.8. $\operatorname{Ext}^i(\Delta(\lambda), \nabla(\mu)) \cong K$ *iff $\lambda = \mu$ and $i = 0$, and otherwise* 0.

Proof. We have an exact sequence $0 \to \operatorname{Hom}(L(\lambda), \nabla(\lambda)) \to \operatorname{Hom}(\Delta(\lambda), \nabla(\lambda)) \to \operatorname{Hom}(\operatorname{Ker}(\Delta(\lambda) \twoheadrightarrow L(\lambda)), \nabla(\lambda))$. Here the simple constituents of the kernel are $L(\nu)$ $(\nu < \lambda)$, and $\operatorname{Hom}(L(\nu), \nabla(\lambda)) = 0$ for $\nu < \lambda$ since $\operatorname{soc}\nabla(\lambda) \cong L(\lambda)$. Thus the last term of the sequence above vanishes. Also, $\operatorname{Hom}(L(\lambda), \nabla(\lambda)) \cong \operatorname{End}(L(\lambda)) \cong \operatorname{Hom}(P(\lambda), L(\lambda)) \cong K$ since $\operatorname{soc}\nabla(\lambda) \cong L(\lambda)$ and $\operatorname{hd} P(\lambda) \cong L(\lambda)$. Thus $\operatorname{Hom}(\Delta(\lambda), \nabla(\lambda)) \cong K$.

We show the vanishings of the other extensions.

- $i = 0$: $\operatorname{Hom}(\Delta(\lambda), \nabla(\mu)) \neq 0$ implies that $\operatorname{Hom}(L(\nu), \nabla(\mu)) \neq 0$ for some $\nu \leq \lambda$. But since $\operatorname{soc}\nabla(\mu) \cong L(\mu)$ this means that $\mu = \nu \leq \lambda$. Thus $\operatorname{Hom}(\Delta(\lambda), \nabla(\mu)) \neq 0$ implies $\mu \leq \lambda$. By the same argument (using $\operatorname{hd}\Delta(\lambda) \cong L(\lambda)$ instead of $\operatorname{soc}\nabla(\mu) \cong L(\mu)$) we see that $\operatorname{Hom}(\Delta(\lambda), \nabla(\mu)) \neq 0$ also implies $\mu \geq \lambda$. Thus $\operatorname{Hom}(\Delta(\lambda), \nabla(\mu)) \neq 0$ implies $\lambda = \mu$.

- $i = 1$: Note that $\operatorname{Ext}^1 = \operatorname{Ext}^1_{\mathcal{C}_{\Lambda'}}$ for any Λ' since $\mathcal{C}_{\Lambda'}$ is closed under extensions. If $\lambda \leq \mu$ then $\Delta(\lambda) \in \mathcal{C}_{\leq\mu}$ and thus $\operatorname{Ext}^1(\Delta(\lambda), \nabla(\mu)) = \operatorname{Ext}^1_{\mathcal{C}_{\leq\mu}}(\Delta(\lambda), \nabla(\mu)) = 0$ by the injectivity of $\nabla(\mu) \in \mathcal{C}_{\leq\mu}$. Otherwise $\nabla(\mu) \in \mathcal{C}_{\not>\lambda}$ and thus $\operatorname{Ext}^1(\Delta(\lambda), \nabla(\mu)) = \operatorname{Ext}^1_{\mathcal{C}_{\not>\lambda}}(\Delta(\lambda), \nabla(\mu)) = 0$ by the projectivity of $\Delta(\lambda) \in \mathcal{C}_{\not>\lambda}$.

- $i \geq 2$: Follows from the exact sequence $0 = \operatorname{Ext}^{i-1}(P(\lambda), \nabla(\mu)) \to \operatorname{Ext}^{i-1}(\operatorname{Ker}(P(\lambda) \twoheadrightarrow \Delta(\lambda)), \nabla(\mu)) \to \operatorname{Ext}^i(\Delta(\lambda), \nabla(\mu)) \to \operatorname{Ext}^i(P(\lambda), \nabla(\mu)) = 0$ and the downward induction on λ.

□

Since $\operatorname{Hom}(P(\lambda), \nabla(\lambda)) = \operatorname{Hom}(P(\lambda)^{\leq\lambda}, \nabla(\lambda)) = \operatorname{Hom}(\Delta(\lambda), \nabla(\lambda)) \cong K$ we see that $(\nabla(\lambda) : L(\lambda)) = 1$.

1.3.4 Standard Filtration

Definition 1.3.9. A *standard (resp. costandard) filtration* of an object M is a filtration $0 = M_0 \subset M_1 \subset \cdots \subset M_r = M$ such that each of its successive quotients M_i/M_{i-1} are isomorphic to standard (resp. costandard) objects. Let $\mathcal{C}^{\Delta}$ denote the full subcategory of the objects having standard filtrations.

Proposition 1.3.10. *For $M \in \mathcal{C}$ having a standard (resp. costandard) filtration, the number of times $\Delta(\lambda)$ (resp. $\nabla(\lambda)$) appears in (any) standard (resp. costandard) filtration of M is given by* $\dim\operatorname{Hom}(M, \nabla(\lambda))$ *(resp.* $\dim\operatorname{Hom}(\Delta(\lambda), M)$*).*

Proof. This is immediate from Proposition 1.3.8. $\square$

For an M having a standard filtration let $(M : \Delta(\lambda))$ denote the number of times $\Delta(\lambda)$ appears in a standard filtration of M (which does not depend on a choice of filtration by the proposition above).

Proposition 1.3.11. *Let $\Lambda' \subset \Lambda$ be an order ideal and let $\lambda \in \Lambda'$ be a maximal element. Then for $M \in \mathcal{C}^{\Delta}$, $\mathrm{Ker}(M^{\Lambda'} \twoheadrightarrow M^{\Lambda' \smallsetminus \{\lambda\}})$ is a direct sum of copies of $\Delta(\lambda)$.*

Note that the proposition in particular implies that $M^{\Lambda'}$ and $\mathrm{Ker}(M \twoheadrightarrow M^{\Lambda'})$ have standard filtrations for any order ideal $\Lambda' \subset \Lambda$, or more generally, for any order ideals $\Lambda'' \subset \Lambda' \subset \Lambda$, $\mathrm{Ker}(M^{\Lambda'} \twoheadrightarrow M^{\Lambda''})$ has a standard filtration. Note also that $(\mathrm{Ker}(M^{\Lambda'} \twoheadrightarrow M^{\Lambda''}) : \Delta(\lambda)) = (M : \Delta(\lambda))$ for $\lambda \in \Lambda' \smallsetminus \Lambda''$ and 0 otherwise.

Proof. First note that the head of $\mathrm{Ker}(M^{\Lambda'} \twoheadrightarrow M^{\Lambda' \smallsetminus \{\lambda\}})$ is a direct sum of copies of $L(\lambda)$ by Remark 1.3.2. Thus the projective cover, in $\mathcal{C}_{\Lambda'}$, of this head is a direct sum of some copies of $\Delta(\lambda)$. So it suffices to show that $\mathrm{Ker}(M^{\Lambda'} \twoheadrightarrow M^{\Lambda' \smallsetminus \{\lambda\}})$ is also a projective cover of $\mathrm{hd}\,\mathrm{Ker}(M^{\Lambda'} \twoheadrightarrow M^{\Lambda' \smallsetminus \{\lambda\}})$, i.e. $\mathrm{Ker}(M^{\Lambda'} \twoheadrightarrow M^{\Lambda' \smallsetminus \{\lambda\}})$ is projective in $\mathcal{C}_{\Lambda'}$.

Let $M' = M^{\Lambda'}$, $M'' = M^{\Lambda' \smallsetminus \{\lambda\}}$ and $N = \mathrm{Ker}(M' \twoheadrightarrow M'')$. We want to show that $\mathrm{Ext}^1(N, L(\mu))$ vanish for all $\mu \in \Lambda'$. We have exact sequences $\mathrm{Hom}(N, \nabla(\mu)/L(\mu)) \to \mathrm{Ext}^1(N, L(\mu)) \to \mathrm{Ext}^1(N, \nabla(\mu))$, $\mathrm{Ext}^1(M', \nabla(\mu)) \to \mathrm{Ext}^1(N, \nabla(\mu)) \to \mathrm{Ext}^2(M'', \nabla(\mu))$ and $\mathrm{Hom}(\mathrm{Ker}(M \twoheadrightarrow M'), \nabla(\mu)) \to \mathrm{Ext}^1(M', \nabla(\mu)) \to \mathrm{Ext}^1(M, \nabla(\mu))$. Here

- $\mathrm{Ext}^1(M, \nabla(\mu))$ vanishes by Proposition 1.3.8.
- $\mathrm{Ext}^2(M'', \nabla(\mu))$ vanishes by Proposition 1.3.8, since M'' has a standard filtration by induction on $|\Lambda'|$.
- $\mathrm{Hom}(N, \nabla(\mu)/L(\mu))$ and $\mathrm{Hom}(\mathrm{Ker}(M \twoheadrightarrow M'), \nabla(\mu))$ vanishes by Remark 1.3.2 since the simple constituents of $\nabla(\mu)/L(\mu)$ (resp. $\nabla(\mu)$) are $L(\nu)$ ($\nu < \mu$ (resp. $\nu \leq \mu$)).

And thus $\mathrm{Ext}^1(N, L(\mu)) = 0$ as desired. $\square$

From the proof of Proposition 1.3.11 we get the following corollary:

Corollary 1.3.12. *$M \in \mathcal{C}$ has a standard filtration if and only if $\mathrm{Ext}^1(M, \nabla(\lambda)) = 0$ for all $\lambda \in \Lambda$.* $\square$

By Proposition 1.3.8 and Corollary 1.3.12 we get the followings.

Corollary 1.3.13. *(1) If $M \in \mathcal{C}$ has a standard filtration then so do its direct summands.*

(2) If $0 \to L \to M \to N \to 0$ is an exact sequence in $\mathcal{C}$ and $M, N \in \mathcal{C}^{\Delta}$, then $L \in \mathcal{C}^{\Delta}$.

Proof. (1):This is clear since if $M = M' \oplus M''$ then $\mathrm{Ext}^1(M, \nabla(\lambda)) \cong \mathrm{Ext}^1(M', \nabla(\lambda)) \oplus \mathrm{Ext}^1(M'', \nabla(\lambda))$.

(2):By Proposition 1.3.8 we have $\mathrm{Ext}^1(M, \nabla(\lambda)) = 0$ and $\mathrm{Ext}^2(N, \nabla(\lambda)) = 0$ for any $\lambda \in \Lambda$. Thus by the exact sequence $\mathrm{Ext}^1(M, \nabla(\lambda)) \to \mathrm{Ext}^1(L, \nabla(\lambda)) \to \mathrm{Ext}^2(N, \nabla(\lambda))$ we see $\mathrm{Ext}^1(L, \nabla(\lambda)) = 0$. Thus by Corollary 1.3.12 we see $L \in \mathcal{C}^{\Delta}$. $\square$

1.3.5 Tilting Objects

Definition 1.3.14. An object $T \in \mathcal{C}$ is called a *tilting* or a *tilting object* if it has a standard filtration and $\mathrm{Ext}^1(\Delta(\lambda), T) = 0$ for all $\lambda \in \Lambda$.

Note that if T is a tilting then so are its direct summands, because $T \in \mathcal{C}$ is a tilting if and only if $\mathrm{Ext}^1(T, \nabla(\lambda))$ and $\mathrm{Ext}^1(\Delta(\lambda), T)$ vanish for all λ.

For $M \in \mathcal{C}^\Delta$, define $\operatorname{supp} M \subset \Lambda$ as the order ideal of Λ generated by all λ such that $(M : \Delta(\lambda)) \neq 0$. For an $X \subset \Lambda$ let X° be the set of all non-maximal elements in X.

Lemma 1.3.15. *Let $M \in \mathcal{C}^\Delta$. Then there is a tilting T and an injection $M \hookrightarrow T$ such that* $\operatorname{supp} T = \operatorname{supp} M$*, $T/M \in \mathcal{C}^\Delta$ and* $\operatorname{supp}(T/M) \subset (\operatorname{supp} M)^\circ$.

Proof. For an $M \in \mathcal{C}^\Delta$, define $\operatorname{def} M \subset \Lambda$, the *defect* of M, to be the order ideal of Λ generated by $\{\lambda \in \Lambda : \mathrm{Ext}^1(\Delta(\lambda), M) \neq 0\}$. Note that $M \in \mathcal{C}^\Delta$ is a tilting if and only if $\operatorname{def} M = \varnothing$.

If $\operatorname{def} M = \varnothing$ then we are done. Assume $\operatorname{def} M \neq \varnothing$. We embed M into an $\tilde{M} \in \mathcal{C}^\Delta$ with strictly smaller defect.

Take a maximal element $\lambda \in \operatorname{def} M$. Then $\mathrm{Ext}^1(\Delta(\lambda), M) \neq 0$, and thus there exists a nonsplit exact sequence $0 \to M \to M_1 \to \Delta(\lambda) \to 0$. Assume $\mathrm{Ext}^1(\Delta(\mu), M_1) \neq 0$ for some $\mu \in \Lambda$. Since there is an exact sequence $\mathrm{Ext}^1(\Delta(\mu), M) \to \mathrm{Ext}^1(\Delta(\mu), M_1) \to \mathrm{Ext}^1(\Delta(\mu), \Delta(\lambda))$ it follows that either $\mathrm{Ext}^1(\Delta(\mu), M) \neq 0$ or $\mathrm{Ext}^1(\Delta(\mu), \Delta(\lambda)) \neq 0$. The first one implies $\mu \in \operatorname{def} M$, while the second one implies $\mu < \lambda$ by Corollary 1.3.6. The latter case implies $\mu \in \operatorname{def} M$ and thus $\mu \in \operatorname{def} M$ in either case. This shows $\operatorname{def} M_1 \subset \operatorname{def} M$. Moreover we claim that $\dim \mathrm{Ext}^1(\Delta(\lambda), M_1) < \dim \mathrm{Ext}^1(\Delta(\lambda), M)$. In fact, we have an exact sequence $\mathrm{Hom}(\Delta(\lambda), M_1) \to \mathrm{Hom}(\Delta(\lambda), \Delta(\lambda)) \to \mathrm{Ext}^1(\Delta(\lambda), M) \to \mathrm{Ext}^1(\Delta(\lambda), M_1) \to \mathrm{Ext}^1(\Delta(\lambda), \Delta(\lambda))$ where the last term is zero by Corollary 1.3.6. But here $\mathrm{Hom}(\Delta(\lambda), M_1) \to \mathrm{Hom}(\Delta(\lambda), \Delta(\lambda))$ is a zero map: otherwise we would have a morphism $\Delta(\lambda) \to M_1$ such that the composition $\Delta(\lambda) \to M_1 \twoheadrightarrow \Delta(\lambda)$ is nonzero and thus an isomorphism (since $\mathrm{End}(\Delta(\lambda)) \cong K$), which contradicts to the assumption that $M_1 \twoheadrightarrow \Delta(\lambda)$ is nonsplit. Thus we have an exact sequence $0 \to \mathrm{End}(\Delta(\lambda)) \to \mathrm{Ext}^1(\Delta(\lambda), M) \to \mathrm{Ext}^1(\Delta(\lambda), M_1) \to 0$ and this shows the claim.

Repeating the construction above we have an $\tilde{M} \in \mathcal{C}^\Delta$ and $M \hookrightarrow \tilde{M}$ such that $\operatorname{def} \tilde{M} \subset \operatorname{def} M \setminus \{\lambda\}$. Repeating again we get an embedding $M \hookrightarrow T$ into a tilting. It is clear from the construction that $T/M \in \mathcal{C}^\Delta$.

We claim that $\operatorname{supp}(T/M) \subset (\operatorname{supp} M)^\circ$. By the construction it suffices to show that $\operatorname{def} M \subset (\operatorname{supp} M)^\circ$. Assume $\mathrm{Ext}^1(\Delta(\lambda), M) \neq 0$ for some λ. Then $\mathrm{Ext}^1(\Delta(\lambda), \Delta(\mu)) \neq 0$ for some $\mu \in \operatorname{supp} M$. Since $\lambda < \mu$ by Corollary 1.3.6, this shows that λ is a non-maximal element in $\operatorname{supp} M$. This shows the claim. ∎

By the lemma there is an embedding $\Delta(\lambda) \hookrightarrow T$ such that T is a tilting, $\operatorname{supp} T = \{\mu : \mu \leq \lambda\}$ and $(T : \Delta(\lambda)) = 1$. So there is an indecomposable summand $T(\lambda)$ of T such that $(T(\lambda) : \Delta(\lambda)) = 1$. By Proposition 1.3.11 we see that there in fact is an embedding $\Delta(\lambda) \hookrightarrow T(\lambda)$ such that $T(\lambda)/\Delta(\lambda)$ has a standard filtration.

Note that λ can be recovered from $T(\lambda)$ as the unique maximal element in $\operatorname{supp} T(\lambda)$: in particular $T(\lambda) \ncong T(\mu)$ if $\lambda \neq \mu$.

Lemma 1.3.16. *Every tilting is a direct sum of the objects $T(\lambda)$.*

Proof. Let $T \neq 0$ be a tilting. Take a maximal element $\lambda \in \operatorname{supp} T$. We show that there is a split surjection $T \twoheadrightarrow T(\lambda)$: this inductively shows the claim.

By the maximality of λ we see $(T : \Delta(\lambda)) \neq 0$. This implies, by Proposition 1.3.11 and the maximality of λ, that there is an injection $\Delta(\lambda) \hookrightarrow T$ with cokernel $T/\Delta(\lambda)$ having a standard filtration. We name the morphisms $\Delta(\lambda) \hookrightarrow T(\lambda)$ and $\Delta(\lambda) \hookrightarrow T$ as f and g respectively.

We have exact sequences $\operatorname{Hom}(T, T(\lambda)) \to \operatorname{Hom}(\Delta(\lambda), T(\lambda)) \to \operatorname{Ext}^1(T/\Delta(\lambda), T(\lambda)) = 0$ and $\operatorname{Hom}(T(\lambda), T) \to \operatorname{Hom}(\Delta(\lambda), T) \to \operatorname{Ext}^1(T(\lambda)/\Delta(\lambda), T) = 0$. Thus there are morphisms $h : T \to T(\lambda)$ and $k : T(\lambda) \to T$ such that $f = hg$ and $g = kf$. Then $f = (hk)^n f$ for any $n \geq 0$, and thus $hk \in \operatorname{End}(T(\lambda))$ is not nilpotent. Then by Fitting's lemma hk is an isomorphism. Thus h is a split surjection, as desired. □

Also, repeated use of Lemma 1.3.15 shows the following:

Lemma 1.3.17. *Any $M \in \mathcal{C}^\Delta$ has a finite resolution $0 \to M \to T_0 \to \cdots \to T_r \to 0$ by tiltings.* □

1.3.6 Ringel Duality

Let us fix a tilting object T such that every indecomposable tilting occurs at least once as its direct summand (such an object is called a *full tilting*). Let $\mathcal{C}^\vee$ be the category of all finite-dimensional left $\operatorname{End}(T)$-modules. Let $F = \operatorname{Hom}(-, T) : \mathcal{C} \to (\mathcal{C}^\vee)^{\mathrm{op}}$.

Note that, since $\operatorname{Ext}^1(N, T) = 0$ for $N \in \mathcal{C}^\Delta$, the functor F is exact on $\mathcal{C}^\Delta$, that is, it maps an exact sequence $0 \to L \to M \to N \to 0$ with $L, M, N \in \mathcal{C}^\Delta$ to an exact sequence $0 \to FN \to FM \to FL \to 0$. This observation implies a more general consequence: suppose that there is an exact sequence $\cdots \to M_1 \to M_0 \to 0$ in $\mathcal{C}^\Delta$ bounded from right. Then Corollary 1.3.13 implies that $\operatorname{Ker}(M_1 \to M_0) \in \mathcal{C}^\Delta$ and thus we can work inductively to see that $0 \to FM_0 \to FM_1 \to \cdots$ is exact.

Lemma 1.3.18. *For any $M \in \mathcal{C}$ and any tilting T', the map $\operatorname{Hom}_{\mathcal{C}}(M, T') \to \operatorname{Hom}_{\mathcal{C}^\vee}(FT', FM)$ induced from F is an isomorphism.*

Proof. For $T' = T$ it is clear. For a general case, it can be seen from the fact that T' appears as a direct summand of some $T^{\oplus m}$ ($m \gg 0$). □

Lemma 1.3.19. *The indecomposable projectives in $\mathcal{C}^\vee$ are given by $FT(\lambda)$ ($\lambda \in \Lambda$).*

Proof. Since $\operatorname{End}(T)$ is, as a left $\operatorname{End}(T)$-module, a direct sum of the modules of the form $FT(\lambda)$ ($\lambda \in \Lambda$), it suffices to show that they are indeed indecomposable. By the previous lemma $\operatorname{End}(FT(\lambda)) \cong \operatorname{End}(T(\lambda))$, and since $T(\lambda)$ is indecomposable $\operatorname{End}(T(\lambda))$ contains no idempotents. Thus $FT(\lambda)$ is indecomposable. □

Proposition 1.3.20. *For $M, N \in \mathcal{C}^\Delta$ and any $i \geq 0$, $\operatorname{Ext}^i(M, N) \cong \operatorname{Ext}^i(FN, FM)$. For $i = 0$ this isomorphism is equal to the map induced from F, and for $i = 1$ this isomorphism is equal to the map $[0 \to N \to X \to M \to 0] \mapsto [0 \to FM \to FX \to FN \to 0]$ where these exact sequences are seen as elements in certain Ext^1 groups.*

Proof. Take a finite tilting resolution $0 \to N \to T_0 \to \cdots \to T_r \to 0$ of N which exists by Lemma 1.3.17. Then $0 \to FT_r \to \cdots \to FT_0 \to FN \to 0$ is a projective resolution since F is exact on $\mathcal{C}^\Delta$ and FT_i are projective. By the same argument as in [24, Theorem 2.7.6] we see (since $\mathrm{Hom}(-, T_i)$ are exact on $\mathcal{C}^\Delta$) that $\mathrm{Ext}^i(M, N)$ is the i-th cohomology of the complex $\mathrm{Hom}(M, T_\bullet)$. On the other hand, $\mathrm{Ext}^i(FN, FM)$ is the i-th cohomology of $\mathrm{Hom}(FT_\bullet, FM)$. By Lemma 1.3.18 the map induced from F gives an isomorphism between these two complexes and thus the first claim follows. The latter claim for $i = 0$ also follows from this argument.

Recall the correspondence from extensions to Ext group ([1, §A.5]): for a projective resolution $\cdots \to P_1 \to P_0 \to M \to 0$, there always exist $f : P_1 \to N$ and $g : P_0 \to X$ such that

$$\begin{tikzcd}
 & P_1 \arrow[r] \arrow[d, "f"] & P_0 \arrow[r] \arrow[d, "g"] & M \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & N \arrow[r] & X \arrow[r] & M \arrow[r] & 0
\end{tikzcd}$$

commutes, and then the element $[0 \to N \to X \to M \to 0] \in \mathrm{Ext}^1(M, N)$ is given by taking the class of $f \in \mathrm{Hom}(P_1, N)$. Chasing the double-complex argument above we see that the correspondence can also be obtained by taking $h : X \to T_0$ and $k : M \to T_1$ such that

$$\begin{tikzcd}
0 \arrow[r] & N \arrow[r] \arrow[d, equal] & X \arrow[r] \arrow[d, "h"] & M \arrow[r] \arrow[d, "k"] & 0 \\
0 \arrow[r] & N \arrow[r] & T_0 \arrow[r] & T_1 &
\end{tikzcd}$$

commute and taking the class of $k \in \mathrm{Hom}(M, T_1)$. Applying F to the diagram above we get

$$\begin{tikzcd}
0 & FN \arrow[l] & FX \arrow[l] & FM \arrow[l] & 0 \arrow[l] \\
0 & FN \arrow[l] \arrow[u, equal] & FT_0 \arrow[l] \arrow[u, "Fh"] & FT_1 \arrow[l] \arrow[u, "Fk"] &
\end{tikzcd}$$

with rows exact and FT_0, FT_1 projective. Thus $[0 \to FM \to FX \to FN \to 0] \in \mathrm{Ext}^1(FN, FM)$ is equal to the class of $Fk \in \mathrm{Hom}(FT_1, FM)$ and this shows the claim. $\square$

Proposition 1.3.21. *$\mathcal{C}^\vee$ is a highest weight category with weight poset Λ^{op}, the opposite poset of Λ, and standard objects $\{F\Delta(\lambda)\}$.*

Proof. Since $\mathrm{Hom}(F\Delta(\lambda), F\Delta(\mu)) \cong \mathrm{Hom}(\Delta(\mu), \Delta(\lambda))$ the first two axioms are clear.

We have an exact sequence $0 \to \Delta(\lambda) \to T(\lambda) \to M \to 0$ such that M has a filtration by $\Delta(\mu)$ $(\mu < \lambda)$. Applying F we get an exact sequence $0 \to FM \to FT(\lambda) \to F\Delta(\lambda) \to 0$ with FM having a filtration by $F\Delta(\mu)$ $(\mu < \lambda)$. This checks the last axiom. $\square$

Proposition 1.3.22. *F restricts to a contravariant equivalence between $\mathcal{C}^\Delta$ and $(\mathcal{C}^\vee)^\Delta$.*

Proof. We saw that $F|_{\mathcal{C}^\Delta}$ is fully faithful and thus it suffices to show the essential-surjectivity: i.e. we want to show that for any $N \in (\mathcal{C}^\vee)^\Delta$ there exists an $M \in \mathcal{C}^\Delta$ such that $FM \cong N$. This follows by the induction on the length of $N \in (\mathcal{C}^\vee)^\Delta$: if $0 \to N' \to N \to N'' \to 0$ is an exact sequence with $N' \cong FM'$ and $N'' \cong FM''$ $(M', M'' \in \mathcal{C}^\Delta)$, then since $\mathrm{Ext}^1(N'', N') \cong \mathrm{Ext}^1(M', M'')$ there is an exact sequence $0 \to M'' \to M \to M' \to 0$ mapped to the above sequence under F, and in particular $N \cong FM$. $\square$

2 Kraśkiewicz-Pragacz modules and highest weight categories

In this section, we define a struture of highest weight category on the category of $\mathfrak{b}$-modules so that the standard objects are KP modules. This enables us to derive a criterion for a $\mathfrak{b}$-module to have KP filtrations in terms of Ext groups with costandard objects in this category, which turn out to be the linear duals of KP modules shifted by some weight. From this it also follows that the category of modules having KP filtrations is closed under taking direct sum components as well as the kernels of surjections.

2.1 Presentation of KP modules by generators and relations

For $w \in S_\infty^{(n)}$ and $1 \leq i < j \leq n$, let $C_{ij}(w) = \{k : (i,k) \notin I(w), (j,k) \in I(w)\} = \{k : k > j, w(i) < w(k) < w(j)\}$ and let $m_{ij}(w) = |C_{ij}(w)| = \#\{k > j : w(i) < w(k) < w(j)\}$ (in particular, $m_{ij}(w) = 0$ if $w(i) > w(j)$). Since $e_{ij}^2 u_w^{(k)} = 0$ for $k \in C_{ij}(w)$ and $e_{ij} u_w^{(k)} = 0$ for $k \notin C_{ij}(w)$, we see that $e_{ij}^{m_{ij}(w)+1}$ annihilates $u_w = u_w^{(1)} \otimes u_w^{(2)} \otimes \cdots$. Let I_w denote the left ideal of $\mathcal{U}(\mathfrak{b})$ generated by $h - \langle \mathrm{code}(w), h\rangle$ $(h \in \mathfrak{h})$ and $e_{ij}^{m_{ij}(w)+1}$ $(i < j)$. Then, by the observation above and the fact that u_w has weight $\mathrm{code}(w)$, there is a unique surjective morphism of $\mathcal{U}(\mathfrak{b})$-modules from $\mathcal{U}(\mathfrak{b})/I_w$ to $\mathcal{S}_w$ sending 1 mod I_w to u_w. The main result in this subsection is the following:

Theorem 2.1.1. *The surjection $\mathcal{U}(\mathfrak{b})/I_w \twoheadrightarrow \mathcal{S}_w$ above is an isomorphism.*

Remark 2.1.2. It is also possible to define u_D and $\mathcal{S}_D$ for a general finite subset $D \subset \{1, \ldots, n\} \times \mathbb{Z}_{>0}$ as in the same way we defined KP modules ($\mathcal{S}_D$ is often called the *flagged Schur module* associated to D, see eg. [16, §7]; the equivalence of the definition there and our definition can be checked by the same argument as in [11, Remark 1.6]). Again in this setting, if we let $m_{ij}(D) = \#\{k : (i,k) \notin D, (j,k) \in D\}$ and $\lambda_i = \#\{k : (i,k) \in D\}$, then $e_{ij}^{m_{ij}(D)+1}$ $(i < j)$ and $h - \langle \lambda, h\rangle$ $(h \in \mathfrak{h})$ annihilate u_D, and therefore we have a surjective morphism $\mathcal{U}(\mathfrak{b})/I_D \twoheadrightarrow \mathcal{S}_D$ where I_D is the left ideal generated by these elements. But this is not an isomorphism for general D: for example, if $D = \{(2,1),(3,2)\}$, then $\mathrm{ch}(\mathcal{U}(\mathfrak{b})/I_D) = x_2x_3 + x_1x_3 + x_2^2 + 2x_1x_2 + x_1^2 + x_1x_2^2x_3^{-1}$ while $\mathrm{ch}(\mathcal{S}_D) = x_2x_3 + x_1x_3 + x_2^2 + 2x_1x_2 + x_1^2$.

The theorem can be reduced to the following lemma, which will be proved below:

Lemma 2.1.3. *Let $w \in S_\infty^{(n)} \smallsetminus \{\mathrm{id}\}$ and take $j, i_1, \ldots, i_A$ and $v, w^{(1)}, \ldots, w^{(A)}$ as in Proposition 1.1.2. Let $x_a = e_{i_a j}^{m_{i_a j}(v)+1}$ for $a = 1, \ldots, A$. Let $I^{(0)} = I_w$ and $I^{(a)} = I^{(a-1)} + \mathcal{U}(\mathfrak{b})x_a$ for $a = 1, \ldots, A$. Also let I'_v be the left ideal of $\mathcal{U}(\mathfrak{b})$ generated by $h - \langle \mathrm{code}(w), h\rangle = h - \langle \mathrm{code}(v) + \epsilon_j, h\rangle$ $(h \in \mathfrak{h})$ and $e_{ij}^{m_{ij}(v)+1}$ $(i < j)$, so $\mathcal{U}(\mathfrak{b})/I'_v \cong \mathcal{U}(\mathfrak{b})/I_v \otimes K_{\epsilon_j}$. Then $I'_v \subset I^{(A)}$ and $I_{w^{(a)}} x_a \subset I^{(a-1)}$ for $a = 1, \ldots, A$.*

Here first we prove Theorem 2.1.1 assuming Lemma 2.1.3. Let $d_w = \dim \mathcal{U}(\mathfrak{b})/I_w$. The conclusion of Lemma 2.1.3 claims that there exist surjective morphisms

$\mathcal{U}(\mathfrak{b})/I_v \otimes K_{\epsilon_j} \cong \mathcal{U}(\mathfrak{b})/I'_v \twoheadrightarrow \mathcal{U}(\mathfrak{b})/I^{(A)} : (x \bmod I'_v) \mapsto (x \bmod I^{(A)})$ and $\mathcal{U}(\mathfrak{b})/I_{w^{(a)}} \twoheadrightarrow I^{(a)}/I^{(a-1)} : (x \bmod I_{w^{(a)}}) \mapsto (xx_a \bmod I^{(a-1)})$ (note that $xx_a \in I^{(a)}$ since $x_a \in I^{(a)}$). Thus $\mathcal{U}(\mathfrak{b})/I_w = \mathcal{U}(\mathfrak{b})/I^{(0)}$ has a quotient filtration $\mathcal{U}(\mathfrak{b})/I^{(0)} \twoheadrightarrow \mathcal{U}(\mathfrak{b})/I^{(1)} \twoheadrightarrow \cdots \twoheadrightarrow \mathcal{U}(\mathfrak{b})/I^{(A)} \twoheadrightarrow 0$ with each subquotient being a quotient of $\mathcal{U}(\mathfrak{b})/I_{w^{(1)}}, \cdots, \mathcal{U}(\mathfrak{b})/I_{w^{(A)}}$ and $\mathcal{U}(\mathfrak{b})/I_v \otimes K_{\epsilon_j}$ respectively. Therefore $d_w \leq d_{w^{(1)}} + \cdots + d_{w^{(A)}} + d_v$. So, by Proposition 1.1.2 and induction on the lexicographic ordering of $(\ell(w), \mathfrak{S}_w(1))$, we see that $d_w \leq \mathfrak{S}_w(1)$ hold for any w. But on the other hand, we have a surjection $\mathcal{U}(\mathfrak{b})/I_w \twoheadrightarrow \mathcal{S}_w$ and thus $d_w \geq \dim \mathcal{S}_w = \mathfrak{S}_w(1)$. Thus $d_w = \mathfrak{S}_w(1)$ and the surjection above must be an isomorphism. This completes the proof of Theorem 2.1.1.

The rest of this subsection is dedicated to the proof of Lemma 2.1.3.

Proof of Lemma 2.1.3. Throughout this proof, let $w \in S_\infty^{(n)} \smallsetminus \{\mathrm{id}\}$ and take $j, i_1, \ldots, i_A, v, w^{(1)}, \ldots, w^{(A)}$ as in Proposition 1.1.2. Take $x_1, \ldots, x_A$ and $I^{(0)}, \ldots, I^{(A)}$ as in Lemma 2.1.3. Let $m_{pq} = m_{pq}(v)$ for $1 \leq p < q \leq n$. For $x, y, \ldots, z \in \mathcal{U}(\mathfrak{b})$, let $\langle x, y, \ldots, z\rangle$ denote the left ideal of $\mathcal{U}(\mathfrak{b})$ generated by $x, y, \ldots, z$.

To make the calculations simple, we use the following basic fact from the representation theory of semisimple Lie algebras:

Proposition 2.1.4. *Let $\mathfrak{n}_3^+ = Ke_{12} \oplus Ke_{13} \oplus Ke_{23}$ be the Lie algebra of all 3×3 strictly upper triangular matrices which acts on $K^3 = Ku_1 \oplus Ku_2 \oplus Ku_3$ and $\bigwedge^2 K^3 = K(u_1 \wedge u_2) \oplus K(u_1 \wedge u_3) \oplus K(u_2 \wedge u_3)$ in the usual way. Then for $a, b \geq 0$, the $\mathcal{U}(\mathfrak{n}_3^+)$-module generated by $(u_2 \wedge u_3)^a \otimes u_3^b \in S^a(\bigwedge^2 K^3) \otimes S^b(K^3)$ ($S^\bullet$ denotes the symmetric product) is isomorphic to $\mathcal{U}(\mathfrak{n}_3^+)/I_{a,b}$ where $I_{a,b}$ is the left ideal of $\mathcal{U}(\mathfrak{n}_3^+)$ generated by e_{12}^{a+1} and e_{23}^{b+1}.*

Proof. First note that $(u_2 \wedge u_3)^a \otimes u_3^b$ is a lowest weight vector of an irreducible representation of $\mathfrak{sl}_3$: i.e. $\mathcal{U}(\mathfrak{n}_3^+)((u_2 \wedge u_3)^a \otimes u_3^b)$ is an irreducible representation of $\mathfrak{sl}_3$. Thus the claim is merely a well-known fact that a finite-dimensional irreducible representation $V(\lambda)$, with lowest weight λ, of a finite-dimensional semisimple Lie algebra $\mathfrak{g}$ with simple root system Δ and upper-triangular part $\mathfrak{n}^+$ is isomorphic to $\mathcal{U}(\mathfrak{n}^+)/\langle e_\alpha^{\langle \lambda, h_\alpha\rangle+1}\rangle_{\alpha \in \Delta}$ as $\mathcal{U}(\mathfrak{n}^+)$-modules ([8, Theorem 21.4]). $\square$

From this proposition, we have the following:

Lemma 2.1.5. *Let $f(x, y, z)$ be a polynomial (in non-commutative variables) and let $a, b \geq 0$. If $f(e_{12}, e_{13}, e_{23})((u_2 \wedge u_3)^a \otimes u_3^b) = 0$, then for $1 \leq p < q < r \leq n$, $f(e_{pq}, e_{pr}, e_{qr}) \in \langle e_{pq}^{a+1}, e_{qr}^{b+1}\rangle$.*

Proof. From Proposition 2.1.4 we have $f(e_{12}, e_{13}, e_{23}) \in \mathcal{U}(\mathfrak{n}_3^+)e_{12}^{a+1} + \mathcal{U}(\mathfrak{n}_3^+)e_{23}^{b+1}$, i.e. $f(e_{12}, e_{13}, e_{23}) = g(e_{12}, e_{13}, e_{23})e_{12}^{a+1} + h(e_{12}, e_{13}, e_{23})e_{23}^{b+1}$ for some g and h. Then $f(e_{pq}, e_{pr}, e_{qr}) = g(e_{pq}, e_{pr}, e_{qr})e_{pq}^{a+1} + h(e_{pq}, e_{pr}, e_{qr})e_{qr}^{b+1} \in \langle e_{pq}^{a+1}, e_{qr}^{b+1}\rangle$. $\square$

With this lemma in hand, it is easy to prove the following:

Lemma 2.1.6. *For $1 \leq p < q < r \leq n$ and $N, M, N', M' \geq 0$,*

(1) $e_{pr}^N e_{qr}^M \equiv 0 \pmod{\langle e_{pq}^{N'+1}, e_{qr}^{M'+1}\rangle}$ *if $N + M > N' + M'$.*

(2) $e_{pq}^N e_{pr}^M \equiv 0 \pmod{\langle e_{pq}^{N'+1}, e_{qr}^{M'+1}\rangle}$ *if $N + M > N' + M'$.*

(3) $e_{pr}^N \equiv \frac{(-1)^N}{N!} e_{qr}^N e_{pq}^N \pmod{\langle e_{pq}^{M+1}, e_{qr}\rangle}$ *(and in fact* $\bmod \langle e_{qr}\rangle$*, although we do not need it here).*

(4) $e_{pr}^N \equiv \frac{1}{N!} e_{pq}^N e_{qr}^N \pmod{\langle e_{pq}, e_{qr}^{M+1}\rangle}$ *(and* $\bmod \langle e_{pq}\rangle$*: we do not need it here).*

(5) $e_{pq}^{N+M+1} e_{qr}^M \equiv 0 \pmod{\langle e_{pq}^{N+1}, e_{qr}^{M+1}\rangle}$.

(6) $e_{pq}^N e_{qr}^M \equiv 0 \pmod{\langle e_{pq}, e_{pr}^N, e_{qr}^{M+1}\rangle}$.

Proof. (1)-(5) follows from straightforward calculations checking the condition of Lemma 2.1.5. (6) also follows from Lemma 2.1.5, since $e_{12}^N e_{23}^M u_3^M = (\text{const.}) \cdot u_1^N u_2^{M-N} = (\text{const.}) \cdot e_{23}^{M-N} e_{13}^N u_3^M$ so $e_{pq}^N e_{qr}^M - (\text{const.}) \cdot e_{qr}^{M-N} e_{pr}^N \in \langle e_{pq}, e_{qr}^{M+1}\rangle$. □

Let us move on to the proof of Lemma 2.1.3. First we prove $I'_v \subset I^{(A)}$. Since $h - \langle \text{code}(w), h\rangle \in I_w \subset I^{(A)}$, it suffices to show $e_{pq}^{m_{pq}+1} \in I^{(A)}$ for all $1 \le p < q \le n$. If $q \ne j$, we have $m_{pq} = m_{pq}(w)$ so $e_{pq}^{m_{pq}+1} \in I_w \subset I^{(A)}$. If $q = j$ and $v(p) > v(j)$, then $m_{pq} = 0 = m_{pq}(w)$ (note that, by the choice of k, there does not exist $r > j$ such that $w(k) < w(r) < w(j)$), and thus again $e_{pq}^{m_{pq}+1} \in I_w \subset I^{(A)}$. If $q = j$ and $p = i_a$, we have $e_{i_a j}^{m_{i_a j}+1} = x_a \in I^{(a)} \subset I^{(A)}$. Otherwise (i.e. if $q = j$, $v(p) < v(j)$ and $p \ne i_1, \dots, i_A$), the conclusion follows from the following lemma:

Lemma 2.1.7. *Let $p < j$, $v(p) < v(j)$ and $p \ne i_1, \dots, i_A$. Then*

(1) There exists some $a \in \{1, \dots, A\}$ such that $v(i_a) > v(p)$.

(2) Let $a \in \{1, \dots, A\}$ be the maximal index such that $v(i_a) > v(p)$. Then $e_{pj}^{m_{pj}+1} \in I^{(a)}$.

Proof. (1): By the assumptions we have $\ell(vt_{pj}) > \ell(v)+1$, and thus there exists an i such that $p < i < j$ and $v(p) < v(i) < v(j)$. Take i to be maximal among such. Then there does not exist i' such that $i < i' < j$ and $v(i) < v(i') < v(j)$, and thus $\ell(vt_{ij}) = \ell(v) + 1$. Therefore i is in $\{i_1, \dots, i_A\}$. This shows (1) since $v(i) > v(p)$.

(2): Let $i = i_a$. Note that $i > p$ by the argument in (1). First we claim that there exists no r such that $i < r < j$ and $v(p) < v(r) < v(i)$. Suppose such r exists. Take r to be maximal among such. Then by the same argument as in (1) we see that r is in $\{i_1, \dots, i_A\}$, and since $r > i$ we have $r = i_b$ for some $b > a$. This contradicts to the choice of a.

From the claim we see $m_{pi} = \#\{r > i : v(p) < v(r) < v(i)\} = \#\{r > j : v(p) < v(r) < v(i)\} = m_{pj} - m_{ij}$. So by Lemma 2.1.6(1), $e_{pj}^{m_{pj}+1} \in \langle e_{pi}^{m_{pi}+1}, e_{ij}^{m_{ij}+1}\rangle$. Since $e_{pi}^{m_{pi}+1} \in I_w \subset I^{(a)}$ and $e_{ij}^{m_{ij}+1} = x_a \in I^{(a)}$ we are done. □

Let us now prove $I_{w^{(a)}} x_a \subset I^{(a-1)}$ $(a = 1, \dots, A)$. Fix $a \in \{1, \dots, A\}$ and let $i = i_a$. We want to prove $(h - \langle \text{code}(w^{(a)}), h\rangle) x_a \in I^{(a-1)}$ for all $h \in \mathfrak{h}$ and $e_{pq}^{m_{pq}(w^{(a)})+1} x_a \in I^{(a-1)}$ for all $p < q$. We first check $(h - \langle \text{code}(w^{(a)}), h\rangle) x_a \in I^{(a-1)}$, i.e., the element $x_a \bmod I^{(a-1)} \in \mathcal{U}(\mathfrak{b})/I^{(a-1)}$ has weight $\text{code}(w^{(a)})$. It is easy to see that $\text{code}(w^{(a)}) = \text{code}(v) + (m_{ij}+1)\epsilon_i - m_{ij}\epsilon_j = \text{code}(w) +$

$(m_{ij}+1)(\epsilon_i-\epsilon_j)$. On the other hand, $x_a \bmod I^{(a-1)} = e_{ij}^{m_{ij}+1} \bmod I^{(a-1)}$ has weight $\mathrm{code}(w)+(m_{ij}+1)(\epsilon_i-\epsilon_j)$ since $1 \bmod I^{(a-1)}$ has weight $\mathrm{code}(w)$ and e_{ij} shifts the weight by $\epsilon_i-\epsilon_j$. This shows the claim.

We now check $e_{pq}^{m_{pq}(w^{(a)})+1}x_a = e_{pq}^{m_{pq}(w^{(a)})+1}e_{ij}^{m_{ij}+1} \in I^{(a-1)}$ for all $1 \le p < q \le n$, case by case. First note that, by Lemma 2.1.7 and the consideration before that lemma, $e_{pq}^{m_{pq}+1} \in I^{(a-1)}$ unless $q=j$ and $v(p) \le v(i)$, and in such case we see $e_{pq}^{m_{pq}+2} = e_{pq}^{m_{pq}(w)+1} \in I_w \subset I^{(a-1)}$. Also note that there does not exist an r such that $i<r<j$ and $v(i)<v(r)<v(j)$, since $vt_{ij} \gtrdot v$.

- $q>j$: In this case we have $m_{pq}(w^{(a)}) = 0 = m_{pq}(w)$, since both w and $w^{(a)}$ are increasing from $(j+1)$-th position and thus there are no $r>q$ with $w(r)<w(q)$ or $w^{(a)}(r)<w^{(a)}(q)$. If $p \ne j$, $e_{pq}e_{ij}^{m_{ij}+1} = e_{ij}^{m_{ij}+1}e_{pq} \in I^{(a-1)}$ since $e_{pq} \in I^{(a-1)}$. If $p=j$, $e_{jq}e_{ij}^{m_{ij}+1} = e_{ij}^{m_{ij}+1}e_{jq} - (m_{ij}+1)e_{ij}^{m_{ij}}e_{iq} \in I^{(a-1)}$ since $e_{jq}, e_{iq} \in I^{(a-1)}$.

- $p=i$ and $q=j$: Trivial from $m_{ij}(w^{(a)})=0$ and $e_{ij}^{m_{ij}(w^{(a)})+1}e_{ij}^{m_{ij}+1} = e_{ij}^{m_{ij}+2} \in I^{(a-1)}$.

  Hereafter we assume $p<q\le j$ and $(p,q) \ne (i,j)$.

- $\{p,q\} \cap \{i,j\} = \varnothing$: If $m_{pq}(w^{(a)}) = m_{pq}$ the proof is trivial since in this case $e_{pq}^{m_{pq}(w^{(a)})+1} \in I^{(a-1)}$ and $e_{pq}^{m_{pq}(w^{(a)})+1}e_{ij}^{m_{ij}+1} = e_{ij}^{m_{ij}+1}e_{pq}^{m_{pq}(w^{(a)})+1}$.

  Consider the case $m_{pq}(w^{(a)}) \ne m_{pq}$. Then:

  - $v(p)<v(q)$ must hold since otherwise $m_{pq}(w^{(a)}) = 0 = m_{pq}$,
  - q must be larger than i, since otherwise $\{w^{(a)}(r) : r>q, w^{(a)}(p) < w^{(a)}(r) < w^{(a)}(q)\} = \{v(r) : r>q, v(p)<v(r)<v(q)\}$ because $w^{(a)}$ and v only differ at i-th and j-th positions, and
  - exactly one of $v(i)$ and $v(j)$ must lie between $v(p)$ and $v(q)$ since otherwise $\{r>q : w^{(a)}(p) < w^{(a)}(r) < w^{(a)}(q)\} = \{r>q : v(p) < v(r) < v(q)\}$.

  Since $i<q<j$ and $\ell(vt_{ij}) = \ell(v)+1$, the case $v(p)<v(i)<v(q)<v(j)$ cannot occur. So $v(i)<v(p)<v(j)<v(q)$. Then we have $p<i$ by the same reason. So we have $p<i<q<j$ and $v(i)<v(p)<v(j)<v(q)$.

  Here $m_{pq}(w^{(a)}) = m_{pq}-1$. Using the fact that there exists no $i<r<j$ with $v(i)<v(r)<v(j)$, we obtain $m_{iq}-m_{ij} = \#\{r>q : v(j) \le v(r) < v(q)\} = m_{pq}-m_{pj}$.

  We have $e_{pq}^{m_{pq}}e_{ij}^{m_{ij}+1} \equiv \frac{(-1)^{m_{ij}+1}}{(m_{ij}+1)!}e_{pq}^{m_{pq}}e_{qj}^{m_{ij}+1}e_{iq}^{m_{ij}+1} \pmod{I^{(a-1)}}$ by Lemma 2.1.6(3) since $e_{qj}, e_{iq}^{m_{iq}+1} \in I^{(a-1)}$. Using $[e_{pq},e_{qj}] = e_{pj}$ and $[e_{pq},e_{pj}] = [e_{qj},e_{pj}] = 0$ we see that the RHS is a linear combination of $e_{qj}^{m_{ij}+1-\nu}e_{pq}^{m_{pq}-\nu}e_{pj}^{\nu}e_{iq}^{m_{ij}+1}$ $(\nu \ge 0)$. Thus it suffices to show that these elements are in $I^{(a-1)}$ for each ν. If $\nu > m_{pj}$ it is clear since $[e_{pj},e_{iq}] = 0$ and $e_{pj}^{m_{pj}+1} \in I^{(a-1)}$. Otherwise, it suffices to show $e_{pq}^{m_{pq}-\nu}e_{iq}^{m_{ij}+1} \in I^{(a-1)}$ since $[e_{pq},e_{pj}] = 0$. This follows from $e_{pq}^{m_{pq}-m_{pj}} e_{iq}^{m_{ij}+1} = e_{pq}^{m_{iq}-m_{ij}} e_{iq}^{m_{ij}+1} \in I^{(a-1)}$, which can be deduced from $e_{pi}, e_{iq}^{m_{iq}+1} \in I^{(a-1)}$ using Lemma 2.1.6(1).

- $p = i$: Since $i < q < j$, the case $v(i) < v(q) < v(j)$ cannot occur. If $v(q) < v(i)$, we have $w^{(a)}(q) < w^{(a)}(i)$ and thus $m_{iq}(w^{(a)}) = 0$. Therefore $e_{iq}^{m_{iq}(w^{(a)})+1} e_{ij}^{m_{ij}+1} = e_{iq} e_{ij}^{m_{ij}+1} = e_{ij}^{m_{ij}+1} e_{iq} \in I^{(a-1)}$ since $e_{iq} \in I^{(a-1)}$. If $v(q) > v(j)$, then $m_{iq}(w^{(a)}) = m_{iq} - m_{ij} - 1$ since $\{r > q : w^{(a)}(i) < w^{(a)}(r) < w^{(a)}(q)\} = \{r > q : v(i) < v(r) < v(q)\} \smallsetminus (\{r > q : v(i) < v(r) < v(j)\} \cup \{j\}) = \{r > q : v(i) < v(r) < v(q)\} \smallsetminus (\{r > j : v(i) < v(r) < v(j)\} \cup \{j\})$, and so we want to show $e_{iq}^{m_{iq}-m_{ij}} e_{ij}^{m_{ij}+1} \in I^{(a-1)}$. This follows from Lemma 2.1.6(2) since $e_{iq}^{m_{iq}+1}, e_{qj} \in I^{(a-1)}$.

- $q = i$: Here we have three cases to consider. If $v(p) < v(i)$, we have $m_{pi}(w^{(a)}) = m_{pi} + m_{ij} + 1$ since $\{r > i : w^{(a)}(p) < w^{(a)}(r) < w^{(a)}(i)\} = \{r > i : v(p) < v(r) < v(i)\} \cup \{r > i : v(i) < v(r) < v(j)\} \cup \{j\} = \{r > i : v(p) < v(r) < v(i)\} \cup \{r > j : v(i) < v(r) < v(j)\} \cup \{j\}$, and so we want to show $e_{pi}^{m_{pi}+m_{ij}+2} e_{ij}^{m_{ij}+1} \in I^{(a-1)}$. This follows from Lemma 2.1.6(5) since $e_{pi}^{m_{pi}+1}, e_{ij}^{m_{ij}+2} \in I^{(a-1)}$. If $v(i) < v(p) < v(j)$, we have $m_{pi}(w^{(a)}) = m_{pj}$ since $\{r > i : w^{(a)}(p) < w^{(a)}(r) < w^{(a)}(i)\} = \{r > i : v(p) < v(r) < v(j)\} = \{r > j : v(p) < v(r) < v(j)\}$ and so we want to show $e_{pi}^{m_{pj}+1} e_{ij}^{m_{ij}+1} \in I^{(a-1)}$. This follows from Lemma 2.1.6(6) since $e_{pi}, e_{ij}^{m_{ij}+2}, e_{pj}^{m_{pj}+1} \in I^{(a-1)}$. Finally if $v(p) > v(j)$, we have $w^{(a)}(p) > w^{(a)}(i)$, $m_{pi}(w^{(a)}) = 0$ and so we want to show $e_{pi} e_{ij}^{m_{ij}+1} \in I^{(a-1)}$. This follows from $e_{pi} e_{ij}^{m_{ij}+1} = e_{ij}^{m_{ij}+1} e_{pi} + (m_{ij}+1) e_{ij}^{m_{ij}} e_{pj}$ and $e_{pi}, e_{pj} \in I^{(a-1)}$.

- $q = j$: This case consists of four subcases (note that the case $p > i$ and $v(i) < v(p) < v(j)$ does not occur):
  - $p < i$ and $v(p) < v(i)$: Here $m_{pj}(w^{(a)}) = m_{pj} - m_{ij}$ since $\{r > j : w^{(a)}(p) < w^{(a)}(r) < w^{(a)}(j)\} = \{r > j : v(p) < v(r) < v(j)\} \smallsetminus \{r > j : v(i) < v(r) < v(j)\}$. So we want to show $e_{pj}^{m_{pj}-m_{ij}+1} e_{ij}^{m_{ij}+1} \in I^{(a-1)}$. If there is no r such that $i < r < j$ and $v(p) < v(r) < v(i)$, then $m_{pi} = m_{pj} - m_{ij}$, and thus $e_{pj}^{m_{pj}-m_{ij}+1} e_{ij}^{m_{ij}+1} = e_{pj}^{m_{pi}+1} e_{ij}^{m_{ij}+1} \in I^{(a-1)}$ by Lemma 2.1.6(1) since $e_{pi}^{m_{pi}+1}, e_{ij}^{m_{ij}+2} \in I^{(a-1)}$. If there exists such r, take r to be the largest among such ones. Then $m_{pr} = m_{pj} - m_{rj}$, since there exists no s such that $r < s < j$ and $v(p) < v(s) < v(r)$. By $e_{ir}, e_{rj}^{m_{rj}+2} \in I^{(a-1)}$ and Lemma 2.1.6(4), we have $e_{pj}^{m_{pj}-m_{ij}+1} e_{ij}^{m_{ij}+1} \equiv \frac{1}{(m_{ij}+1)!} e_{pj}^{m_{pj}-m_{ij}+1} e_{ir}^{m_{ij}+1} e_{rj}^{m_{ij}+1} \pmod{I^{(a-1)}}$. Since the elements $e_{pr}^{m_{pr}+1} = e_{pr}^{m_{pj}-m_{rj}+1}$ and $e_{rj}^{m_{rj}+2}$ are in $I^{(a-1)}$ we see from Lemma 2.1.6(1) that $e_{pj}^{m_{pj}-m_{ij}+1} e_{rj}^{m_{ij}+1} \in I^{(a-1)}$. Thus $e_{pj}^{m_{pj}-m_{ij}+1} e_{ij}^{m_{ij}+1} e_{rj}^{m_{ij}+1} = e_{ij}^{m_{ij}+1} e_{pj}^{m_{pj}-m_{ij}+1} e_{rj}^{m_{ij}+1} \in I^{(a-1)}$ and this shows the claim.
  - $p < i$ and $v(p) > v(i)$: Here $m_{pj}(w^{(a)}) = 0$ since $w^{(a)}(p) > w^{(a)}(j)$. Thus $e_{pj}^{m_{pj}(w^{(a)})+1} e_{ij}^{m_{ij}+1} = e_{pj} e_{ij}^{m_{ij}+1} \in I^{(a-1)}$ by $e_{pi}, e_{ij}^{m_{ij}+2} \in I^{(a-1)}$ and Lemma 2.1.6(1).

– $p > i$ and $v(p) < v(i)$: Here $m_{pj}(w^{(a)}) = m_{pj} - m_{ij}$ since $\{r > j : w^{(a)}(p) < w^{(a)}(r) < w^{(a)}(j)\} = \{r > j : v(p) < v(r) < v(j)\} \setminus \{r > j : v(i) < v(r) < v(j)\}$. Thus $e_{pj}^{m_{pj}(w^{(a)})+1} e_{ij}^{m_{ij}+1} = e_{pj}^{m_{pj}-m_{ij}+1} e_{ij}^{m_{ij}+1} \in I^{(a-1)}$ by $e_{ip}, e_{pj}^{m_{pj}+2} \in I^{(a-1)}$ and Lemma 2.1.6(1).

– $p > i$ and $v(p) > v(j)$: Here $m_{pj}(w^{(a)}) = 0$ since $w^{(a)}(p) > w^{(a)}(j)$. Thus $e_{pj}^{m_{pj}(w^{(a)})+1} e_{ij}^{m_{ij}+1} = e_{pj} e_{ij}^{m_{ij}+1} \in I^{(a-1)}$ since $e_{pj} e_{ij}^{m_{ij}+1} = e_{ij}^{m_{ij}+1} e_{pj}$ and $e_{pj} \in I^{(a-1)}$.

Thus we checked $e_{pq}^{m_{pq}(w^{(a)})+1} x_a \in I^{(a-1)}$ for all $p < q$. This finishes the proof of Lemma 2.1.3. $\square$

Remark 2.1.8. It is clear from the definition that $m_{pr}(w) \leq m_{pq}(w) + m_{qr}(w)$ for any $p < q < r$. If $m_{pr}(w) = m_{pq}(w) + m_{qr}(w)$, then by Lemma 2.1.6(1) we have $e_{pr}^{m_{pr}(w)+1} \in \langle e_{pq}^{m_{pq}(w)+1}, e_{qr}^{m_{qr}(w)+1} \rangle$. Thus in fact the generators $e_{pr}^{m_{pr}(w)+1}$ such that there exists some $q \in \{p+1, \ldots, r-1\}$ with $m_{pr}(w) = m_{pq}(w) + m_{qr}(w)$ are superfluous.

2.2 Projectivity of KP modules

In this subsection, using the presentations of KP modules obtained in the previous subsection, we show a certain projectivity property for KP modules which will be essential in showing the highest weight structure for KP modules.

Let $\mathcal{C}$ be the category of all weight $\mathfrak{b}$-modules. For $\Lambda \subset \mathbb{Z}^n$, let $\mathcal{C}_\Lambda$ be the full subcategory of $\mathcal{C}$ consisting of all weight $\mathfrak{b}$-modules whose weights are in Λ. Note that if $|\Lambda| < \infty$ and $\Lambda' = \{\rho - \lambda : \lambda \in \Lambda\}$ ($\rho = (n-1, n-2, \ldots, 0)$), then $\mathcal{C}_{\Lambda'} \cong \mathcal{C}_\Lambda^{\mathrm{op}}$ by $M \mapsto M^* \otimes K_\rho$ (it is also true for infinite Λ if we take M^* to be the *graded* dual $\bigoplus (M_\lambda)^*$ of M).

Lemma 2.2.1 (cf. [23, Lemma 3.1.1])**.** *For any finite $\Lambda \subset \mathbb{Z}^n$, $\mathcal{C}_\Lambda$ has enough projective objects and enough injective objects.*

Proof. By the duality remarked before the lemma it is sufficient to show the enough-projectivity.

For $\lambda \in \Lambda$, let $P_\lambda = \mathcal{U}(\mathfrak{b})/\langle h - \langle h, \lambda \rangle \rangle_{h \in \mathfrak{h}}$ (which is isomorphic to $\mathcal{U}(\mathfrak{n}^+)$ as a $\mathcal{U}(\mathfrak{n}^+)$-module, by PBW theorem) and let P_λ^Λ be the largest quotient of P_λ which is in $\mathcal{C}_\Lambda$, i.e. P_λ^Λ is the quotient of P_λ by the submodule generated by all weight spaces $(P_\lambda)_\mu$ ($\mu \notin \Lambda$). Then P_λ^Λ is projective in $\mathcal{C}_\Lambda$ since for $N \in \mathcal{C}_\Lambda$, $\mathrm{Hom}(P_\lambda^\Lambda, N) = \mathrm{Hom}(P_\lambda, N) = N_\lambda$.

For a general $M \in \mathcal{C}_\Lambda$, $P_M = \bigoplus_\lambda (P_\lambda^\Lambda)^{\oplus \dim M_\lambda}$ is a projective object in $\mathcal{C}_\Lambda$ and there is a surjection $P_M \twoheadrightarrow M$. This shows the lemma. $\square$

Note that, if $\lambda \in \Lambda$, P_λ^Λ has a unique maximum proper submodule $\bigoplus_{\mu \neq \lambda} (P_\lambda^\Lambda)_\mu$; therefore the head of P_λ^Λ is K_λ, and thus P_λ^Λ is the projective cover of K_λ in $\mathcal{C}_\Lambda$.

We introduce some order relations (other than dominance order) on $\mathbb{Z}^n$ as follows. For two permutations $w, v \in S_\infty$, we write $w \underset{\mathrm{lex}}{\leq} v$ if $w = v$ or there exists an $i \geq 1$ such that $w(j) = v(j)$ for all $j < i$ and $w(i) < v(i)$. Likewise, we write $w \underset{\mathrm{rlex}}{\leq} v$ if $w = v$ or there exists an $i \geq 1$ such that $w(j) = v(j)$ for all $j > i$

and $w(i) < v(i)$. For $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{Z}^n$, define $|\lambda| = \sum \lambda_i$. If $\lambda, \mu \in \mathbb{Z}^n_{\geq 0}$ and $w = \mathrm{perm}(\lambda), v = \mathrm{perm}(\mu)$, we write $\lambda \geq \mu$ if $|\lambda| = |\mu|$ and $w^{-1} \underset{\mathrm{lex}}{\leq} v^{-1}$. For general λ and μ in $\mathbb{Z}^n$, take k so that $\lambda + k\mathbf{1}$ and $\mu + k\mathbf{1}$ are in $\mathbb{Z}^n_{\geq 0}$, and define $\lambda \geq \mu \iff \lambda + k\mathbf{1} \geq \mu + k\mathbf{1}$. Note that this definition does not depend on the choice of k since $\mathrm{perm}(\lambda)^{-1} \underset{\mathrm{lex}}{\leq} \mathrm{perm}(\mu)^{-1} \iff \mathrm{perm}(\lambda + \mathbf{1})^{-1} \underset{\mathrm{lex}}{\leq} \mathrm{perm}(\mu + \mathbf{1})^{-1}$ for $\lambda, \mu \in \mathbb{Z}^n_{\geq 0}$. We define the other ordering $\geq'$ in the same way, except that we use $\underset{\mathrm{rlex}}{\leq}$ instead of $\underset{\mathrm{lex}}{\leq}$: i.e. $\lambda \geq' \mu$ if and only if $\mathrm{perm}(\lambda + k\mathbf{1})^{-1} \underset{\mathrm{rlex}}{\leq} \mathrm{perm}(\mu + k\mathbf{1})^{-1}$ for $k \gg 0$.

We write $\lambda \preceq \mu$ for $\lambda, \mu \in \mathbb{Z}^n$ if both $\lambda \leq \mu$ and $\lambda \leq' \mu$ hold.

Lemma 2.2.2. *For $\lambda, \mu \in \mathbb{Z}^n$, $\lambda \geq \mu$ if and only if $\rho - \lambda \geq' \rho - \mu$.*

Proof. We may assume $|\lambda| = |\mu|$. We only need to prove the "only if" direction since the other implication follows by exchanging λ and μ. Take integers L and M so that $\lambda + L\mathbf{1}, \mu + L\mathbf{1}, \rho - \lambda + M\mathbf{1}, \rho - \mu + M\mathbf{1} \in \mathbb{Z}^n_{\geq 0}$. Let $w = \mathrm{perm}(\lambda + L\mathbf{1}), v = \mathrm{perm}(\mu + L\mathbf{1}), w' = \mathrm{perm}(\rho - \lambda + M\mathbf{1})$ and $v' = \mathrm{perm}(\rho - \mu + M\mathbf{1})$. Then $w, v, w', v' \in S^{(n)}_\infty \cap S_N$ where $N = n + L + M$, and these permutations are related by $w'(i) = N + 1 - w(i), v'(i) = N + 1 - v(i)$ for $i = 1, \ldots, n$. More precisely, we have $w'^{-1}(p) = \begin{cases} w^{-1}(N+1-p) & (w^{-1}(N+1-p) \leq n) \\ n + N + 1 - w^{-1}(N+1-p) & (w^{-1}(N+1-p) > n) \end{cases}$ and $v'^{-1}(p) = \begin{cases} v^{-1}(N+1-p) & (v^{-1}(N+1-p) \leq n) \\ n + N + 1 - v^{-1}(N+1-p) & (v^{-1}(N+1-p) > n) \end{cases}$.

Now assume $w^{-1} \underset{\mathrm{lex}}{\leq} v^{-1}$. if $w = v$ we have nothing to prove so we assume that there is an i such that $w^{-1}(1) = v^{-1}(1), \ldots, w^{-1}(i-1) = v^{-1}(i-1), w^{-1}(i) < v^{-1}(i)$. By the above description of w' and v' it is clear that $w'^{-1}(j) = v'^{-1}(j)$ for $j > N+1-i$. We show $w'^{-1}(N+1-i) < v'^{-1}(N+1-i)$. If $w^{-1}(i) < v^{-1}(i) \leq n$ we have $w'^{-1}(N+1-i) = w^{-1}(i) < v^{-1}(i) = v'^{-1}(N+1-i)$. If $w^{-1}(1) \leq n < v^{-1}(i)$ we have $w'^{-1}(N+1-i) \leq n < v'^{-1}(N+1-i)$. The case $n < w^{-1}(i) < v^{-1}(i)$ cannot occur, since in such case $w^{-1}(i) = n+1+\#\{j < i : w^{-1}(j) > n\}$, $v^{-1}(i) = n+1+\#\{j < i : v^{-1}(j) > n\}$ and $\{j < i : w^{-1}(j) > n\} = \{j < i : v^{-1}(j) > n\}$. Thus we have checked $w'^{-1}(N+1-i) < v'^{-1}(N+1-i)$ and thus $w'^{-1} \underset{\mathrm{rlex}}{\leq} v'^{-1}$. This shows the lemma. □

Lemma 2.2.3. *For $\lambda, \mu \in \mathbb{Z}^n$, if $\min_{1 \leq i \leq n} \lambda_i > \min_{1 \leq i \leq n} \mu_i$ then $\lambda < \mu$.*

Proof. We may assume that $\lambda, \mu \in \mathbb{Z}^n_{\geq 0}$. Let $m = \min_{1 \leq i \leq n} \mu_i$. Then $w = \mathrm{perm}(\lambda)$ and $v = \mathrm{perm}(\mu)$ satisfy $w^{-1}(1) = v^{-1}(1) = n+1, \ldots, w^{-1}(m) = v^{-1}(m) = n+m$ and $w^{-1}(m+1) > n \geq v^{-1}(m+1)$. Thus $w^{-1} \underset{\mathrm{lex}}{>} v^{-1}$. □

By Lemma 2.2.3 we see that $\mathbb{Z}^n_{\geq 0}$ is an order ideal in $(\mathbb{Z}^n, <)$, and by Lemma 2.2.3 and Lemma 2.2.2 we see that $\rho - \mathbb{Z}^n_{\geq 0} = \{\rho - \lambda : \lambda \in \mathbb{Z}^n_{\geq 0}\}$ is an order ideal in $(\mathbb{Z}^n, <')$. So in particular we see that $\Lambda_n = \{(a_1, \ldots, a_n) : 0 \leq a_i \leq n - i\} = \mathbb{Z}^n_{\geq 0} \cap (\rho - \mathbb{Z}^n_{\geq 0})$ is an order ideal in $(\mathbb{Z}^n, \prec)$

The main result of this subsection is the following proposition:

Proposition 2.2.4. *Let $\Lambda \subset (\mathbb{Z}^n, \prec)$ be a finite order ideal containing $\lambda \in \mathbb{Z}^n$ as one of its maximal elements. Then the module $\mathcal{S}_\lambda$ is in $\mathcal{C}_\Lambda$ and gives a projective cover of K_λ in $\mathcal{C}_\Lambda$.*

It is easy to see that the head of $\mathcal{S}_\lambda$ is K_λ. Therefore, to prove Proposition 2.2.4 we have to prove the following four facts: for every $\lambda, \mu \in \mathbb{Z}^n$,

(1) $(\mathcal{S}_\lambda)_\mu \neq 0$ implies $\lambda \geq \mu$,

(2) $(\mathcal{S}_\lambda)_\mu \neq 0$ implies $\lambda \geq' \mu$,

(3) $\mathrm{Ext}^1(\mathcal{S}_\lambda, K_\mu) \neq 0$ implies $\lambda < \mu$ (here Ext^1 is taken in either $\mathcal{C}$ or $\mathcal{C}_{\leq\lambda}$, which does not matter since $\mathcal{C}_{\leq\lambda}$ is closed under extension), and

(4) $\mathrm{Ext}^1(\mathcal{S}_\lambda, K_\mu) \neq 0$ implies $\lambda <' \mu$.

Before starting the proof, first let us make an observation on the weights of $\mathcal{S}_w$ ($w \in S_\infty^{(n)}$). For $j \geq 1$, let $l_j(w) = \#\{i : i < j, w(i) > w(j)\}$ as in the definition of KP modules. Since $\mathcal{S}_w$ is a submodule of $\bigotimes_{j\geq 1} \bigwedge^{l_j(w)} K^{j-1}$, any weight of $\mathcal{S}_w$ is a weight of $\bigotimes_{j\geq 1} \bigwedge^{l_j(w)} K^{j-1}$. The weights of the latter space can be understood as follows. A *w-pattern* (terminology only for here) is a sequence of sets $(I_1, I_2, \ldots)$ such that $I_j \subset \{1, \ldots, j-1\}$ and $|I_j| = l_j(w)$. Define the *weight* $(\mu_1, \mu_2, \ldots)$ of a w-pattern $(I_1, I_2, \ldots)$ by $\mu_i = \#\{j : i \in I_j\}$. Then it is easy to see that μ is a weight of $\bigotimes_{j\geq 1} \bigwedge^{l_j(w)} K^{j-1}$ if and only if it is the weight of some w-pattern.

Let us now prove (1) and (2) above.

(1): We may assume that λ and μ are in $\mathbb{Z}^n_{\geq 0}$, since $(\mathcal{S}_\lambda)_\mu \neq 0 \iff (\mathcal{S}_{\lambda+k\mathbf{1}})_{\mu+k\mathbf{1}} \neq 0$ for any $\lambda, \mu \in \mathbb{Z}^n$ and any $k \in \mathbb{Z}$. Let $w = \mathrm{perm}(\lambda)$ and $v = \mathrm{perm}(\mu)$. We prove a stronger statement: if μ is the weight of some w-pattern $(I_1, I_2, \ldots)$ then $\lambda \geq \mu$.

We first show $w^{-1}(1) \leq v^{-1}(1)$. Let $i = w^{-1}(1)$. Since $w(1), \ldots, w(i-1) > w(i)$ we have $l_i(w) = i-1$, and thus $I_i = \{1, \ldots, i-1\}$. Thus $\mu_1, \ldots, \mu_{i-1} \geq 1$. Since $v^{-1}(1) = \min\{j : \mu_j = 0\}$, this shows $w^{-1}(1) \leq v^{-1}(1)$.

Now consider the case $w^{-1}(1) = v^{-1}(1)$. In this case we have $\mu_i = 0$, i.e. none of the sets I_j contains i. Define $\sigma_i : \mathbb{Z}_{>0} \setminus \{i\} \to \mathbb{Z}_{>0}$ by $\sigma_i(i') = \begin{cases} i' & (i' < i) \\ i'-1 & (i' > i) \end{cases}$, and consider a new sequence of sets $I' = (\sigma_i(I_1), \ldots, \sigma_i(I_{i-1}), \sigma_i(I_{i+1}), \sigma_i(I_{i+2}), \ldots)$. It is easy to check that I' is a w'-pattern with weight $\mathrm{code}(v')$, where $w' = [w(1)-1 \ \cdots \ w(i-1)-1 \ w(i+1)-1 \ w(i+2)-1 \cdots]$ and $v' = [v(1)-1 \ \cdots \ v(i-1)-1 \ v(i+1)-1 \ v(i+2)-1 \cdots]$. An inductive argument shows that $w'^{-1} \underset{\mathrm{lex}}{\leq} v'^{-1}$. This shows $w^{-1} \underset{\mathrm{lex}}{\leq} v^{-1}$. □

(2): We may assume $\lambda, \mu \in \mathbb{Z}^n_{\geq 0}$ as before. Let $w = \mathrm{perm}(\lambda)$ and $v = \mathrm{perm}(\mu)$. We prove a stronger statement: if μ is the weight of some w-pattern $(I_1, I_2, \ldots)$ then $\lambda \geq' \mu$. Take N so that $w, v \in S_N$. Note that $I_{N+1} = I_{N+2} = \cdots = \varnothing$ since $l_w(N+1) = l_w(N+2) = \cdots = 0$.

We first show $w^{-1}(N) \leq v^{-1}(N)$. Let $i = w^{-1}(N)$. Then we have $l_i(w) = 0$ and thus $I_i = \varnothing$. Thus for $j < i$, we have $j \notin I_1, \ldots, I_j, I_i$, and thus $\mu_j \leq N-j-1$. Since $v^{-1}(N) = \min\{i : \mu_i = N-i\}$ this shows $v^{-1}(N) \geq w^{-1}(N)$.

Now consider the case $w^{-1}(N) = v^{-1}(N)$. Then $\mu_i = N-i$. Since $i \notin I_1, \ldots, I_i$ we must have $i \in I_{i+1}, \ldots I_N$. It is easy to see that $I' = (\sigma_i(I_1), \ldots, \sigma_i(I_{i-1}), \sigma_i(I_{i+1} \setminus \{i\}), \ldots, \sigma_i(I_N \setminus \{i\}), \varnothing, \varnothing, \ldots)$ is a w'-pattern with weight $\mathrm{code}(v')$ where $w' = [w(1) \cdots w(i-1)\, w(i+1) \cdots w(N)]$ and $v' =$

$[v(1)\cdots v(i-1)\, v(i+1)\cdots v(N)]$. An inductive argument shows $w'^{-1} \underset{\text{rlex}}{\leq} v'^{-1}$. This shows $w^{-1} \underset{\text{rlex}}{\leq} v^{-1}$. $\square$

For (3) and (4), we need the following observation. By Theorem 2.1.1, for any $w \in S_\infty^{(n)}$ there is a projective resolution of $\mathcal{S}_w$ in $\mathcal{C}$ of the form $\cdots \to P_1 \to P_0 \to \mathcal{S}_w \to 0$ with $P_0 = P_{\mathrm{code}(w)}$ and $P_1 = \bigoplus_{p<q} P_{\mathrm{code}(w)+(m_{pq}(w)+1)(\epsilon_p-\epsilon_q)}$. Here by Remark 2.1.8, we can in fact replace P_1 by a smaller module: sum over all $p<q$ such that

$$(*) : \text{there does not exist } p<r<q \text{ with } m_{pq}(w) = m_{pr}(w) + m_{rq}(w).$$

In particular, $\mathrm{Ext}^1(\mathcal{S}_w, K_\mu) = 0$ unless $\mu = \mathrm{code}(w) + (m_{pq}(w)+1)(\epsilon_p - \epsilon_q)$ for some $p<q$ satisfying the property $(*)$ above.

(3): We may assume that $\lambda, \mu \in \mathbb{Z}^n_{\geq 0}$, since $\lambda < \mu \iff \lambda + k\mathbf{1} < \mu + k\mathbf{1}$ and $\mathrm{Ext}^1(\mathcal{S}_\lambda, K_\mu) \neq 0 \iff \mathrm{Ext}^1(\mathcal{S}_{\lambda+k\mathbf{1}}, K_{\mu+k\mathbf{1}}) \neq 0$ for any $\lambda, \mu \in \mathbb{Z}^n$ and any $k \in \mathbb{Z}$.

Let $w = \mathrm{perm}(\lambda)$ and $v = \mathrm{perm}(\mu)$. By the remark above, we have $\mu = \lambda + (m_{pq}(w)+1)(\epsilon_p - \epsilon_q)$ for some $p<q$ (and therefore $w \neq v$). We first show $w^{-1}(1) \geq v^{-1}(1)$. Let $i = w^{-1}(1)$. If $i < v^{-1}(1)$, then $\mu_i > 0$ while $\lambda_i = 0$, and so $p = i$. But then $m_{pq}(w) = \#\{r > q : w(i) < w(r) < w(q)\} = \#\{r > q : w(r) < w(q)\} = \mathrm{code}(w)_q = \lambda_q$ and so we have $\mu_q = -1$, which contradicts to $\mu \in \mathbb{Z}^n_{\geq 0}$. Therefore $i \geq v^{-1}(1)$.

If $i = v^{-1}(1)$, then $\lambda_i = \mu_i = 0$, and so $p, q \neq i$. Therefore $\lambda' = (\lambda_1 - 1, \ldots, \lambda_{i-1} - 1, \lambda_{i+1}, \lambda_{i+2}, \ldots)$ and $\mu' = (\mu_1 - 1, \ldots, \mu_{i-1} - 1, \mu_{i+1}, \mu_{i+2}, \ldots)$ satisfy $\mu' = \lambda' + (m_{pq}(w)+1)(\epsilon_{p'} - \epsilon_{q'})$ for $p' = \sigma_i(p)$, $q' = \sigma_i(q)$. Moreover, $m_{pq}(w) = m_{p'q'}(w')$, where $w' = [w(1)-1 \,\cdots\, w(i-1)-1\; w(i+1)-1\; w(i+2)-1 \,\cdots]$. Thus an inductive argument shows $w'^{-1} \underset{\text{lex}}{\geq} v'^{-1}$ where $v' = [v(1)-1 \,\cdots\, v(i-1)-1\; v(i+1)-1\; v(i+2)-1 \,\cdots] = \mathrm{perm}(\mu')$. This shows $w^{-1} \underset{\text{lex}}{\geq} v^{-1}$.
$\square$

(4): We may assume that $\lambda, \mu \in \mathbb{Z}^n_{\geq 0}$ as before. Let $w = \mathrm{perm}(\lambda)$, $v = \mathrm{perm}(\mu)$. Take N so that $w, v \in S_N$. We have $\mu = \lambda + (m_{pq}(w)+1)(\epsilon_p - \epsilon_q)$ for some $p<q$ as before, with the property $(*)$ above. We first show $w^{-1}(N) \geq v^{-1}(N)$.

Assume $w^{-1}(N) < v^{-1}(N)$. Then $\lambda_{w^{-1}(N)} = N - w^{-1}(N)$ while $\mu_{w^{-1}(N)} < N - w^{-1}(N)$ and so $q = w^{-1}(N)$.

We first claim that there does not exist r such that $p<r<q$ and $w(p) < w(r)$. Suppose such r exists. Take r to be the largest among such. By the property $(*)$ we have $m_{pq}(w) < m_{pr}(w) + m_{rq}(w)$. This means that there is a column index $1 \leq j \leq N$ such that $(p,j), (q,j) \in I(w), (r,j) \notin I(w)$ or $(p,j), (q,j) \notin I(w), (r,j) \in I(w)$, since other types of columns contribute to LHS and RHS by the same value. We see that neither of these cases can occur as follows.

- Assume the former case. Then $(p,j) \in I(w)$ implies $w(j) < w(p) < w(r)$ and $(q,j) \in I(w)$ implies $j > q > r$. These shows $(r,j) \in I(w)$. Contradiction.
- Assume the latter case. $w(q) = N > w(j)$ and $(q,j) \notin I(w)$ implies $j < q$. Also, $(r,j) \in I(w)$ implies $j > r > p$, and this together with $(p,j) \notin I(w)$

shows $w(p) < w(j)$. Thus j satisfies $p < j < q$, $w(p) < w(j)$ and $j > r$. This contradicts to the choice of r.

Since there does not exist r such that $p < r < q$ and $w(p) < w(r)$, we see that $m_{pq}(w) = \#\{r > q : w(p) < w(r) < w(q)\} = \#\{r > q : w(p) < w(r)\} = N - w(p) - 1 - \#\{r < q : w(p) < w(r)\} = N - w(p) - 1 - \#\{r < p : w(p) < w(r)\}$. From this and $\lambda_p = \mathrm{code}(w)_p = \#\{r > p : w(r) < w(p)\} = w(p) - 1 - \#\{r < p : w(r) < w(p)\}$, we see $\mu_p = \lambda_p + m_{pq}(w) + 1 = N - p$. This means $v^{-1}(N) = \min\{p' : \mu_{p'} = N - p'\} \leq p < q = w^{-1}(N)$. This contradicts to the assumption and thus we see $w^{-1}(N) \geq v^{-1}(N)$.

If $w^{-1}(N) = v^{-1}(N)$, then $p \neq w^{-1}(N) \neq q$ as before, and we can inductively argue in the same way as in (3). □

Remark 2.2.5. It is easy to see that the projective cover of K_λ in $\mathcal{C}_\Lambda$ ($\lambda \in \Lambda \subset \mathbb{Z}^n$) is given by the largest quotient $(P_\lambda)^\Lambda$ of P_λ whose weights are in Λ. Thus from the theorem above we see that $\mathcal{S}_\lambda \cong (P_\lambda)^\Lambda$ for any order ideal $\Lambda \subset (\mathbb{Z}^n, \prec)$ containing λ as one of its maximal elements.

Let $\Lambda \subset (\mathbb{Z}^n, \prec)$ be an order ideal and $\lambda \in \Lambda$ be one of its maximal elements as above. If a weight $\mathfrak{b}$-module M is generated by an element of weight λ then M is a quotient of P_λ. So if in addition $M \in \mathcal{C}_\Lambda$ then it follows that M is in fact a quotient of $\mathcal{S}_\lambda$.

2.3 Highest weight structure

The main result in this subsection is the following:

Theorem 2.3.1. *Let $\Lambda \subset \mathbb{Z}^n$ be a finite order ideal with respect to the ordering $\prec$. Then $\mathcal{C}_\Lambda$ is a highest weight category with the weight poset $(\Lambda, \prec)$ and the standard objects $\{\mathcal{S}_\lambda : \lambda \in \Lambda\}$ (for the definitions of $\mathcal{C}_\Lambda$ and the order relation $\prec$ see the beginning of Section 2.2).*

Proof. We have already verified the first two axioms in the previous subsection. Below we verify that the last axiom holds.

Let $\mu \in \Lambda$. The projective cover of K_μ in $\mathcal{C}_\Lambda$ is P_μ^Λ, the largest quotient of P_μ (see the proof of Lemma 2.2.1) whose weights are all in Λ. We want to show that there exists a surjection $P_\mu^\Lambda \twoheadrightarrow \mathcal{S}_\mu$ whose kernel admits a filtration by modules $\mathcal{S}_\nu$ ($\nu \succ \mu$).

Index the elements of Λ as $\lambda^1, \dots, \lambda^l$ so that $\lambda^i \prec \lambda^j$ implies $i < j$. Let $\Lambda^i = \{\lambda^1, \dots, \lambda^i\}$. Note that Λ^i and $\rho - \Lambda^i = \{\rho - \lambda^1, \dots, \rho - \lambda^i\}$ are order ideals in Λ with respect to the ordering $\prec$, and λ^i and $\rho - \lambda^i$ are their maximal elements respectively. If $\mu = \lambda^k$, then $P_\mu^{\Lambda^k} \cong \mathcal{S}_\mu$ since both $P_\mu^{\Lambda^k}$ and $\mathcal{S}_\mu$ give projective covers of K_μ in the category $\mathcal{C}_{\Lambda^k}$. We show that the kernel of $P_\mu^{\Lambda^i} \twoheadrightarrow P_\mu^{\Lambda^{i-1}}$ is a direct sum of some copies of $\mathcal{S}_{\lambda^i}$ for any i: this shows the claim since $0 \subset \mathrm{Ker}(P_\mu^\Lambda \twoheadrightarrow P_\mu^{\Lambda^{l-1}}) \subset \mathrm{Ker}(P_\mu^\Lambda \twoheadrightarrow P_\mu^{\Lambda^{l-2}}) \subset \cdots \subset \mathrm{Ker}(P_\mu^\Lambda \twoheadrightarrow P_\mu^{\Lambda^k}) = \mathrm{Ker}(P_\mu^\Lambda \twoheadrightarrow \mathcal{S}_\mu)$ gives a desired filtration.

Let $1 \leq i \leq l$. We have a $\mathfrak{b}$-homomorphism $(P_\mu^{\Lambda^i})_{\lambda^i} \otimes P_{\lambda^i}^{\Lambda^i} \to P_\mu^{\Lambda^i}$ (where on the left-hand side tensor product $\mathfrak{b}$ acts only on $P_{\lambda^i}^{\Lambda^i}$) defined by $xu_\mu \otimes yu_{\lambda^i} \mapsto yxu_\mu$ for $x \in \mathcal{U}(\mathfrak{n}^+)_{\lambda^i - \mu}$ and $y \in \mathcal{U}(\mathfrak{n}^+)$ where u_μ is the image of $1 \otimes 1 \in \mathcal{U}(\mathfrak{n}^+) \otimes K_\mu \cong P_\mu \twoheadrightarrow P_\mu^{\Lambda^i}$ (this definition does not depend on the choice of y since the submodule of $P_\mu^{\Lambda^i}$ generated by xu_μ is a quotient of $P_{\lambda^i}^{\Lambda^i}$

by definition). The image of the morphism above is $\mathrm{Ker}(P_\mu^{\Lambda^i} \twoheadrightarrow P_\mu^{\Lambda^{i-1}})$, so it induces a surjection $(P_\mu^{\Lambda^i})_{\lambda^i} \otimes P_{\lambda^i}^{\Lambda^i} \twoheadrightarrow \mathrm{Ker}(P_\mu^{\Lambda^i} \twoheadrightarrow P_\mu^{\Lambda^{i-1}})$. Since the left-hand side is a direct sum of copies of $P_{\lambda^i}^{\Lambda^i} \cong \mathcal{S}_{\lambda^i}$, it is enough to show that this surjection is in fact an isomorphism for any i and any μ.

We want to show that the surjection $(P_\mu^{\Lambda^i})_{\lambda^i} \otimes (P_{\lambda^i}^{\Lambda^i})_\nu \twoheadrightarrow \mathrm{Ker}((P_\mu^{\Lambda^i})_\nu \twoheadrightarrow (P_\mu^{\Lambda^{i-1}})_\nu)$, obtained by restricting the surjection above to ν-weight spaces, is an isomorphism for any i and any μ, ν. Since the claim does not contain any information on Λ we may take Λ to be sufficiently large: to be precise, we assume that $\Lambda \supset \{\kappa : \mu \trianglelefteq \kappa \trianglelefteq \nu\}$. This in particular implies that $(P_\mu^\Lambda)_\nu \cong \mathcal{U}(\mathfrak{n}^+)_{\nu-\mu}$ as vector spaces.

We have a quotient filtration $(P_\mu^{\Lambda^l})_\nu \twoheadrightarrow (P_\mu^{\Lambda^{l-1}})_\nu \twoheadrightarrow \cdots \twoheadrightarrow 0$ of vector spaces, and by the argument above its successive quotients are quotients of $(P_\mu^{\Lambda^i})_{\lambda^i} \otimes (P_{\lambda^i}^{\Lambda^i})_\nu$. Thus we have $\dim(P_\mu^\Lambda)_\nu \le \sum_{i=1}^l \dim((P_\mu^{\Lambda^i})_{\lambda^i} \otimes (P_{\lambda^i}^{\Lambda^i})_\nu)$. If we show that the equality holds then the desired isomorphism $(P_\mu^{\Lambda^i})_{\lambda^i} \otimes (P_{\lambda^i}^{\Lambda^i})_\nu \cong \mathrm{Ker}((P_\mu^{\Lambda^i})_\nu \twoheadrightarrow (P_\mu^{\Lambda^{i-1}})_\nu)$ follows for all i.

We know $(P_{\lambda^i}^{\Lambda^i})_\nu \cong (\mathcal{S}_{\lambda^i})_\nu$ by Proposition 2.2.4. Now consider $(P_\mu^{\Lambda^i})_{\lambda^i}$. Since $P_\mu^{\Lambda^i}$ is the quotient of P_μ by the submodule generated by all weight spaces $(P_\mu)_\sigma$ ($\sigma \notin \Lambda^i$), we have a vector space isomorphism

$$(P_\mu^{\Lambda^i})_{\lambda^i} \cong \mathcal{U}(\mathfrak{n}^+)_{\lambda^i-\mu}/\mathrm{Span}_K\{xy : x \in \mathcal{U}(\mathfrak{n}^+)_{\lambda^i-\sigma},\, y \in \mathcal{U}(\mathfrak{n}^+)_{\sigma-\mu} \text{ for some } \sigma \notin \Lambda^i\}.$$

The algebra antiautomorphism on $\mathcal{U}(\mathfrak{n}^+)$ given by $X \mapsto -X$ ($X \in \mathfrak{n}^+$) induces an isomorphism between this space and

$$\begin{aligned}&\mathcal{U}(\mathfrak{n}^+)_{\lambda^i-\mu}/\mathrm{Span}_K\{yx : x \in \mathcal{U}(\mathfrak{n}^+)_{\lambda^i-\sigma},\, y \in \mathcal{U}(\mathfrak{n}^+)_{\sigma-\mu} \text{ for some } \sigma \notin \Lambda^i\}\\ &= \mathcal{U}(\mathfrak{n}^+)_{\lambda^i-\mu}/\mathrm{Span}_K\{yx : x \in \mathcal{U}(\mathfrak{n}^+)_{\lambda^i-\sigma},\, y \in \mathcal{U}(\mathfrak{n}^+)_{\sigma-\mu} \text{ for some } \sigma \text{ s.t. } \rho-\sigma \notin \rho-\Lambda^i\}\end{aligned}$$

where $\rho - \Lambda^i = \{\rho - \sigma : \sigma \in \Lambda^i\}$. By the same argument as above we see that this is isomorphic to $(P_{\rho-\lambda^i}^{\rho-\Lambda^i})_{\rho-\mu}$. By Proposition 2.2.4 we see $(P_{\rho-\lambda^i}^{\rho-\Lambda^i})_{\rho-\mu} \cong (\mathcal{S}_{\rho-\lambda^i})_{\rho-\mu}$. Thus, after all, we see that $(P_\mu^{\Lambda^i})_{\lambda^i} \cong (\mathcal{S}_{\rho-\lambda^i})_{\rho-\mu}$.

Since $(P_\mu^{\Lambda^i})_{\lambda^i} \cong (\mathcal{S}_{\rho-\lambda^i})_{\rho-\mu}$ and $(P_{\lambda^i}^{\Lambda^i})_\nu \cong (\mathcal{S}_{\lambda^i})_\nu$ as we have seen above, we see that $\dim((P_\mu^{\Lambda^i})_{\lambda^i} \otimes (P_{\lambda^i}^{\Lambda^i})_\nu)$ is equal to the coefficient of $x^{\rho-\mu}y^\nu$ in $\mathfrak{S}_{\rho-\lambda^i}(x)\mathfrak{S}_{\lambda^i}(y)$. So $\sum_{i=1}^l \dim((P_\mu^{\Lambda^i})_{\lambda^i} \otimes (P_{\lambda^i}^{\Lambda^i})_\nu)$ is equal to the coefficient of $x^{\rho-\mu}y^\nu$ in $\sum_{i=1}^l \mathfrak{S}_{\rho-\lambda^i}(x)\mathfrak{S}_{\lambda^i}(y) = \sum_{\lambda\in\Lambda} \mathfrak{S}_{\rho-\lambda}(x)\mathfrak{S}_\lambda(y)$. Since we have assumed $\Lambda \supset \{\kappa : \mu \trianglelefteq \kappa \trianglelefteq \nu\}$ this is equal to the coefficient of $x^{\rho-\mu}y^\nu$ in $\sum_{\lambda\in\mathbb{Z}^n} \mathfrak{S}_{\rho-\lambda}(x)\mathfrak{S}_\lambda(y)$. On the other hand as we have remarked above $(P_\mu^\Lambda)_\nu \cong \mathcal{U}(\mathfrak{n}^+)_{\nu-\mu}$. Thus the proof of the theorem is now reduced to the following elementary lemma:

Lemma 2.3.2. *For $\mu, \nu \in \mathbb{Z}^n$, $\dim \mathcal{U}(\mathfrak{n}^+)_{\nu-\mu}$ is equal to the coefficient of $x^{\rho-\mu}y^\nu$ in $\sum_{\kappa\in\mathbb{Z}^n} \mathfrak{S}_{\rho-\kappa}(x)\mathfrak{S}_\kappa(y)$.*

Let us prove this lemma. We use the following result from [19]:

Lemma 2.3.3 ([19, Lemma 6.2 and Corollary 9.2, reformulated])**.** *For a positive integer N, define a bilinear form $\langle\cdot,\cdot\rangle$ on $\mathbb{Z}[x_1^{\pm1}, \dots, x_N^{\pm1}]$ by $\langle x^\alpha, x^\beta\rangle = \delta_{\alpha\beta}$. Then for $w, v \in S_N$, $\langle \mathfrak{S}_w, \mathfrak{S}_{w_0 v}(x_1^{-1}, \dots, x_N^{-1}) \prod_{1\le i<j\le N}(x_i - x_j)\rangle = \delta_{wv}$, where $w_0 = [N\ N-1\ \cdots\ 1] \in S_N$.*

We slightly modify this lemma into a form which is more suitable for our use:

Lemma 2.3.4. *If we define a bilinear form $\langle\cdot,\cdot\rangle$ on $\mathbb{Z}[x_1^{\pm 1},\dots,x_n^{\pm 1}]$ by $\langle x^\alpha, x^\beta\rangle = \delta_{\alpha\beta}$, then for $\lambda,\mu\in\mathbb{Z}^n$, $\langle \mathfrak{S}_\lambda, \mathfrak{S}_{\rho-\mu}(x_1^{-1},\dots,x_n^{-1})\prod_{1\le i<j\le n}(x_i-x_j)\rangle = \delta_{\lambda\mu}$.*

Proof. We may assume that $\lambda,\mu\in\mathbb{Z}^n_{\ge 0}$. Let $w=\mathrm{perm}(\lambda), v=\mathrm{perm}(\mu)$. Take N so that $w,v\in S_N$. Then by the previous lemma, we have

$$\langle \mathfrak{S}_w, \mathfrak{S}_{w_0v}(x_1^{-1},\dots,x_N^{-1})\prod_{1\le i<j\le N}(x_i-x_j)\rangle = \delta_{wv} = \delta_{\lambda\mu} \quad \cdots(*)$$

where $w_0=[N\ N-1\ \cdots\ 1]\in S_N$.

Since $\prod_{1\le i<j\le N}(x_i-x_j) = \prod_{i\le n<j}(x_i-x_j)\cdot\prod_{i<j\le n}(x_i-x_j)\cdot\prod_{n<i<j}(x_i-x_j)$, we see that

$$\begin{aligned}\prod_{1\le i<j\le N}(x_i-x_j) &\equiv (x_1\cdots x_n)^{N-n}\cdot\prod_{i<j\le n}(x_i-x_j)\cdot\prod_{n<i<j}(x_i-x_j)\\ &= (x_1\cdots x_n)^{N-n}\cdot\prod_{i<j\le n}(x_i-x_j)\cdot(x_{n+1}^{N-n-1}x_{n+2}^{N-n-2}\cdots x_{N-1}+R)\end{aligned}$$

modulo terms whose degree with respect to the variables $x_{n+1},\dots,x_N$ is strictly greater than $T=\binom{N-n}{2}$, where R is some homogeneous polynomial in $x_{n+1},\dots,x_N$ of degree T and without monomial $x_{n+1}^{N-n-1}x_{n+2}^{N-n-2}\cdots x_{N-1}$.

Let f be the sum of all terms in $\mathfrak{S}_{w_0v}$ whose degree in $x_{n+1},\dots,x_N$ is equal to T. Note that, since $\mathfrak{S}_{w_0v}$ is a linear combination of monomials $x_1^{a_1}\cdots x_n^{a_n}$ $(0\le a_i\le N-i)$, the degree of its terms with respect to variables $x_{n+1},\dots,x_N$ are always at most T: that is, $\mathfrak{S}_{w_0v}=f+$(terms with degree $<T$ in variables $x_{n+1},\dots,x_N$). Also note $f\in x_{n+1}^{N-n-1}\cdots x_{N-1}\mathbb{Z}[x_1,\dots,x_n]$ by the same reason. We claim $f=(x_1\cdots x_n)^{N-n}x_{n+1}^{N-n-1}\cdots x_{N-1}\mathfrak{S}_{\rho-\mu}$.

Let $w_{n,N}=[1\ \cdots\ n\ N\ N-1\ \cdots\ n+1]\in S_N$. Note that $w_{n,N}w_0v\in S_\infty^{(n)}$, $\mathrm{code}(w_{n,N}w_0v)=\rho-\mu+(N-n)\mathbf{1}$ and thus $\mathfrak{S}_{w_{n,N}w_0v}=(x_1\cdots x_n)^{N-n}\mathfrak{S}_{\rho-\mu}$. We have $\mathfrak{S}_{w_{n,N}w_0v}=\partial_{w_{n,N}}\mathfrak{S}_{w_0v}$, where $\partial_{w_{n,N}}=(\partial_{n+1}\partial_{n+2}\cdots\partial_{N-1})\cdot(\partial_{n+2}\cdots\partial_{N-1})\cdot\cdots\cdot\partial_{N-1}$. Since the operators ∂_i $(n+1\le i\le N-1)$ lower the degree in variables $x_{n+1},\dots,x_N$ by one, $\partial_{w_{n,N}}$ annihilates $\mathfrak{S}_{w_0v}-f$. Thus $\mathfrak{S}_{w_{n,N}w_0v}=\partial_{w_{n,N}}f$. Since $f\in x_{n+1}^{N-n-1}\cdots x_{N-1}\mathbb{Z}[x_1,\dots,x_n]$ and $\partial_{w_{n,N}}x_{n+1}^{N-n-1}\cdots x_{N-1}=1$ we see that $\partial_{w_{n,N}}f=f/(x_{n+1}^{N-n-1}\cdots x_{N-1})$. Thus $f=x_{n+1}^{N-n-1}\cdots x_{N-1}\mathfrak{S}_{w_{n,N}w_0v}=(x_1\cdots x_n)^{N-n}x_{n+1}^{N-n-1}\cdots x_{N-1}\mathfrak{S}_{\rho-\mu}$. This shows the claim above.

We have seen that

$$\prod_{1\le i<j\le N}(x_i-x_j)\equiv(x_1\cdots x_n)^{N-n}\cdot\prod_{1\le i<j\le n}(x_i-x_j)\cdot(x_{n+1}^{N-n-1}x_{n+2}^{N-n-2}\cdots x_{N-1}+R)$$

and

$$\mathfrak{S}_{w_0v}(x_1^{-1},\dots,x_N^{-1})\equiv(x_1\cdots x_n)^{-N+n}x_{n+1}^{-N+n+1}\cdots x_{N-1}^{-1}\cdot\mathfrak{S}_{\rho-\mu}(x_1^{-1},\cdots,x_n^{-1})$$

modulo terms having degrees $>T$ and $>-T$ in variables $x_{n+1},\dots,x_N$ respectively. Thus $\mathfrak{S}_{w_0v}(x_1^{-1},\dots,x_N^{-1})\prod_{1\le i<j\le N}(x_i-x_j)$ is equal to

$$\mathfrak{S}_{\rho-\mu}(x_1^{-1},\cdots,x_n^{-1})\cdot\prod_{1\le i<j\le n}(x_i-x_j)\cdot(1+x_{n+1}^{-N+n+1}\cdots x_{N-1}^{-1}R)$$

modulo terms with degree > 0 in variables $x_{n+1}, \dots, x_N$. Since the variables $x_{n+1}, \dots, x_N$ do not appear in $\mathfrak{S}_w$ and $x_{N-1}^{-N+n+1} \cdots x_{N-1}^{-1} R$ does not have a constant term, this shows

$$\langle \mathfrak{S}_w, \mathfrak{S}_{w_0 v}(x_1^{-1}, \dots, x_N^{-1}) \prod_{1 \le i < j \le N} (x_i - x_j) \rangle = \langle \mathfrak{S}_w, \mathfrak{S}_{\rho-\mu}(x_1^{-1}, \dots, x_n^{-1}) \prod_{1 \le i < j \le n} (x_i - x_j) \rangle.$$

This, together with $(*)$, finishes the proof of Lemma 2.3.4. $\square$

Let us come back to the proof of Lemma 2.3.2. Essentially this is a "Cauchy formula" for the dual bases $\{\mathfrak{S}_\lambda\}$ and $\{\mathfrak{S}_{\rho-\mu}(x_1^{-1}, \dots, x_n^{-1}) \prod (x_i - x_j)\}$ appeared in Lemma 2.3.4, but since we are dealing with an infinite-dimensional space a careful justification is needed. Let $c_{\alpha\beta}$ be the coefficient of $x^\alpha y^\beta$ in $\sum_{\kappa \in \mathbb{Z}^n} \mathfrak{S}_{\rho-\kappa}(x) \mathfrak{S}_\kappa(y)$. We observe that if $c_{\rho-\mu,\nu} \neq 0$, then there exists some κ such that $\rho - \mu \trianglerighteq \rho - \kappa$ and $\nu \trianglerighteq \kappa$, and so $\nu \trianglerighteq \kappa \trianglerighteq \mu$. Thus $c_{\rho-\mu,\nu} = 0$ for $\nu \not\trianglerighteq \mu$. Using this as the base case, if we show $\sum_{g \in S_n} \operatorname{sgn}(g) c_{\rho-\mu, \nu-\rho+g\rho} = \delta_{\mu\nu}$, then we can show $c_{\rho-\mu,\nu} = \dim \mathcal{U}(\mathfrak{n}^+)_{\nu-\mu}$ by induction on ν since $\sum_\kappa \dim \mathcal{U}(\mathfrak{n}^+)_\kappa x^\kappa = \prod_{i<j} (1 - x_i x_j^{-1})^{-1}$ and $\prod_{i<j}(1 - x_i x_j^{-1}) = \sum_{g \in S_n} \operatorname{sgn}(g) x^{\rho - g\rho}$. We show the equivalent claim $\sum_{g \in S_n} \operatorname{sgn}(g) c_{\alpha, \beta + g\rho} = \delta_{\alpha, -\beta}$.

Since $c_{\alpha,\beta+g\rho} = c_{\alpha+k\mathbf{1}, \beta+g\rho-k\mathbf{1}}$, we may assume that $-\beta \in \mathbb{Z}^n_{\ge 0}$. We may further assume, by replacing α and β by $\alpha + k\mathbf{1}$ and $\beta - k\mathbf{1}$ for a sufficiently large k, that if $\kappa \in \mathbb{Z}^n$ satisfies $\alpha \trianglerighteq \kappa \trianglerighteq -\beta + \rho - g\rho$ for some $g \in S_n$ then $\kappa \in \mathbb{Z}^n_{\ge 0}$ (this is possible by the remark at the end of Section 1.2). We only have to consider the case $|\alpha| = -|\beta|$. Let $d = |\alpha|$. Let V be the space of all (ordinary) polynomials in $x_1, \dots, x_n$ which are homogeneous of degree d. Equip V with a bilinear form $\langle x^\sigma, x^\tau \rangle = \delta_{\sigma\tau}$. Then by Lemma 2.3.4 the bases $\{\mathfrak{S}_\kappa : \kappa \in \mathbb{Z}^n_{\ge 0}, |\kappa| = d\}$ and $\{[\mathfrak{S}_{\rho-\kappa}(x_1^{-1}, \dots, x_n^{-1}) \prod_{1 \le i<j \le n}(x_i - x_j)] : \kappa \in \mathbb{Z}^n_{\ge 0}, |\kappa| = d\}$ of V are dual to each other; here for $f \in \mathbb{Z}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$, $[f]$ is the sum of all terms in f which do not contain any negative powers of $x_1, \dots, x_n$. Thus we have

$$\begin{aligned}
\sum_{\substack{\gamma \in \mathbb{Z}^n_{\ge 0} \\ |\gamma| = d}} x^\gamma y^\gamma &\equiv \sum_{\substack{\kappa \in \mathbb{Z}^n_{\ge 0} \\ |\kappa| = d}} \mathfrak{S}_\kappa(x_1, \dots, x_n) \mathfrak{S}_{\rho-\kappa}(y_1^{-1}, \dots, y_n^{-1}) \prod_{1 \le i < j \le n} (y_i - y_j) \\
&= \left(\sum_{\substack{\kappa \in \mathbb{Z}^n_{\ge 0} \\ |\kappa| = d}} \mathfrak{S}_\kappa(x_1, \dots, x_n) \mathfrak{S}_{\rho-\kappa}(y_1^{-1}, \dots, y_n^{-1}) \right) \left(\sum_{g \in S_n} \operatorname{sgn}(g) y^{g\rho} \right) \qquad \cdots (*)
\end{aligned}$$

modulo terms containing some negative powers of some y_i (we used the fact that for any finite-dimensional vector space V, the sum $\sum \varphi_i \otimes \varphi_i^* \in V \otimes V^*$ does not depend on the choice of dual bases $\{\varphi_i\} \subset V, \{\varphi_i^*\} \subset V^*$). Since $-\beta \in \mathbb{Z}^n_{\ge 0}$, the coefficients of $x^\alpha y^{-\beta}$ are equal for both sides. The coefficient for the LHS is $\delta_{\alpha,-\beta}$. Moreover, if $\kappa \in \mathbb{Z}^n$ and $\mathfrak{S}_\kappa(x_1, \dots, x_n) \mathfrak{S}_{\rho-\kappa}(y_1^{-1}, \dots, y_n^{-1})$ contains some monomial of the form $x^\alpha y^{-\beta - g\rho}$ ($g \in S_n$) with a nonzero coefficient, then such κ must satisfy $\alpha \trianglerighteq \kappa$ and $\beta + g\rho \trianglerighteq \rho - \kappa$ and so $\kappa \in \mathbb{Z}^n_{\ge 0}$. Thus the coefficient of $x^\alpha y^{-\beta}$ in the RHS is the same as the coefficient of $x^\alpha y^{-\beta}$ in $\left(\sum_{\substack{\kappa \in \mathbb{Z}^n \\ |\kappa| = d}} \mathfrak{S}_\kappa(x_1, \dots, x_n) \mathfrak{S}_{\rho-\kappa}(y_1^{-1}, \dots, y_n^{-1}) \right) \left(\sum_{g \in S_n} \operatorname{sgn}(g) y^{g\rho} \right)$. Since this coefficient is $\sum_{g \in S_n} \operatorname{sgn}(g) c_{\alpha, \beta + g\rho}$ we are done. $\square$

Note that, by Proposition 2.2.4 and Lemma 2.2.2, the costandard objects in $\mathcal{C}_\Lambda$ are given by $\mathcal{S}^*_{\rho-\lambda} \otimes K_\rho$ $(\lambda \in \Lambda)$.

From general theory of highest weight categories we obtain the following corollaries:

Corollary 2.3.5. *A finite dimensional weight $\mathfrak{b}$-module M has a KP filtration if and only if $\mathrm{Ext}^1(M, \mathcal{S}^*_{\rho-\lambda} \otimes K_\rho) = 0$ for all $\lambda \in \mathbb{Z}^n$. In such case, the number of times the KP module $\mathcal{S}_\lambda$ $(\lambda \in \mathbb{Z}^n)$ appears in (any) standard filtration of M is given by $\dim \mathrm{Hom}(M, \mathcal{S}^*_{\rho-\lambda} \otimes K_\rho)$.*

Proof. This follows from Proposition 1.3.10 and Corollary 1.3.12 (note that the order ideal generated by the weights of M is finite by Lemma 2.2.3). $\square$

Corollary 2.3.6. *(1) If $M = M_1 \oplus \cdots \oplus M_r$ and M has a KP filtration then so does each M_i.*

(2) If $0 \to L \to M \to N \to 0$ is an exact sequence and M and N have KP filtrations then so does L.

Proof. This follows from Corollary 1.3.13. $\square$

3 Tensor product of Kraśkiewicz-Pragacz modules

3.1 Existence of KP filtrations for tensor products

In this subsection we use the highest weight theory for KP modules developed in the previous section to show the following:

Theorem 3.1.1. *For any $\lambda, \mu \in \mathbb{Z}^n$, the tensor product module $\mathcal{S}_\lambda \otimes \mathcal{S}_\mu$ has a KP filtration.*

In order to prove Theorem 3.1.1, first we need the special cases where one of the KP modules is $\mathcal{S}_{s_i}$, corresponding to Monk's rule for Schubert polynomials.

Proposition 3.1.2. *Let $w \in S_\infty^{(n)}$ and let $1 \leq \nu \leq n-1$. Then $\mathcal{S}_w \otimes \mathcal{S}_{s_\nu}$ has a KP filtration.*

Proof. This is a special case of the Pieri rule for KP modules (Theorem 3.2.4) which we will prove in Section 3.2. Note that the proof given there does not use the result in this subsection. See also Remark 3.2.9. $\square$

Using Proposition 3.1.2 and the highest weight theory for KP modules we have developed above, we can now proceed to a proof for a more general result:

Theorem 3.1.3. *For any $w \in S_n$ and $v \in S_\infty^{(n)}$, $\mathcal{S}_w \otimes \mathcal{S}_v$ has a KP filtration.*

If we let $n \to \infty$, we see that for any $w, v \in S_\infty$, the module (over the Lie algebra $\mathfrak{b}_\infty = \bigcup_n \mathfrak{b}_n$ of upper triangular matrices of infinite size with finitely many nonzero entries) $\mathcal{S}_w \otimes \mathcal{S}_v$ has a filtration by KP modules, if we regard $\mathfrak{b}_n$-modules $\mathcal{S}_w$ ($w \in S_\infty^{(n)}$) as $\mathfrak{b}_\infty$-modules through the morphism $\mathfrak{b}_\infty \twoheadrightarrow \mathfrak{b}_n$ annihilating all e_{ij} with $j > n$. In particular, we see that the theorem in fact holds for any $w, v \in S_\infty^{(n)}$, and since the general KP modules $\mathcal{S}_\lambda$ ($\lambda \in \mathbb{Z}^n$) are just the KP modules $\mathcal{S}_w$ ($w \in S_\infty^{(n)}$) shifted by some weight $k\mathbf{1}$ ($k \in \mathbb{Z}$) this implies Theorem 3.1.1.

In order to prove Theorem 3.1.3, we begin with some observations. For a $w \in S_n$, we define a $\mathfrak{b}$-module $T_w = \bigotimes_{2 \leq i \leq n} \left(\bigwedge^{l_i(w)} K^{i-1} \right)$, where $l_i(w) = \#\{j < i : w(j) > w(i)\}$ as in the definition of KP modules. Since T_w is a direct sum component of $\bigotimes_{2 \leq i \leq n} \bigotimes^{l_i(w)} K^{i-1} = \bigotimes_{2 \leq i \leq n} \bigotimes^{l_i(w)} \mathcal{S}_{s_{i-1}}$, T_w has a KP filtration by Proposition 3.1.2 and Corollary 2.3.6(1). We show the following lemma:

Lemma 3.1.4. *Let $w \in S_n$. Then there exists an exact sequence $0 \to \mathcal{S}_w \to T_w \to N \to 0$ such that N has a filtration whose each subquotient is isomorphic to some $\mathcal{S}_u$ ($u \in S_n$, $u^{-1} \underset{\text{lex}}{>} w^{-1}$).*

We see first that the theorem easily follows from this lemma by a descending induction on the lexicographic order of w^{-1}. From the lemma we get an exact sequence $0 \to \mathcal{S}_w \otimes \mathcal{S}_v \to T_w \otimes \mathcal{S}_v \to N \otimes \mathcal{S}_v \to 0$. Here $T_w \otimes \mathcal{S}_v$ have a KP filtration by Proposition 3.1.2 and Corollary 2.3.6(1), since it is a direct summand of $\left(\bigotimes_{2 \leq i \leq n} \bigotimes^{l_i(w)} \mathcal{S}_{s_{i-1}} \right) \otimes \mathcal{S}_v$. Moreover $N \otimes \mathcal{S}_v$ have a KP filration by the induction hypothesis. Hence the claim follows from Corollary 2.3.6(2).

Let us now prove Lemma 3.1.4. As we have seen in Proposition 2.2.4, for $y, z \in S_\infty^{(n)}$:

- if $(\mathcal{S}_y)_{\mathrm{code}(z)} \neq 0$ (i.e. the coefficient of $x^{\mathrm{code}(z)}$ in $\mathfrak{S}_y$ is nonzero) then $z^{-1} \underset{\mathrm{lex}}{\geq} y^{-1}$, and

- if $\mathrm{Ext}^1(\mathcal{S}_y, K_{\mathrm{code}(z)}) \neq 0$ then $z^{-1} \underset{\mathrm{lex}}{<} y^{-1}$.

In particular, if $w, u \in S_\infty^{(n)}$ and $\mathrm{Ext}^1(\mathcal{S}_w, \mathcal{S}_u) \neq 0$, then there exists a $z \in S_\infty^{(n)}$ such that $(\mathcal{S}_u)_{\mathrm{code}(z)} \neq 0$ and $\mathrm{Ext}^1(\mathcal{S}_w, K_{\mathrm{code}(z)}) \neq 0$, and thus $u^{-1} \underset{\mathrm{lex}}{\leq} z^{-1} \underset{\mathrm{lex}}{<} w^{-1}$.

Proof of Lemma 3.1.4. Let $l_i = l_i(w)$ and let the integers $n_{wu} \in \mathbb{Z}$ be defined by $\prod_{2 \leq i \leq n} e_{l_i}(x_1, \ldots, x_{i-1}) = \sum_{u \in S_n} n_{wu} \mathfrak{S}_u$ where e_k denotes the k-th elementary symmetric polynomial. Since the left-hand side is the character of T_w, the number n_{wu} is the number of times $\mathcal{S}_u$ appears as a subquotient in (any) KP filtration of T_w.

By Proposition 1.1.3 we have $\sum_{u \in S_n} \mathfrak{S}_u(x) \mathfrak{S}_{uw_0}(y) = \prod_{i+j \leq n}(x_i + y_j) = \sum_{0 \leq a_i \leq n-i} \big(\prod_{1 \leq i \leq n-1} x_i^{n-i-a_i} \cdot \prod_{1 \leq i \leq n-1} e_{a_i}(y_1, \ldots, y_{n-i}) \big)$. Thus there exists a bilinear form $\langle , \rangle : H_n \times H_n \to \mathbb{Z}$ such that $\langle \mathfrak{S}_u, \mathfrak{S}_{u'w_0} \rangle = \delta_{uu'}$ and $\langle x^{\rho - \alpha}, \prod_{1 \leq i \leq n-1} e_{\beta_i}(x_1, \ldots, x_{n-i}) \rangle = \delta_{\alpha,\beta}$. Then

$$\begin{aligned} n_{wu} &= \langle \mathfrak{S}_{uw_0}, \prod_{2 \leq i \leq n} e_{l_i}(x_1, \ldots, x_{i-1}) \rangle \\ &= (\text{coefficient of } x_1^{n-1-l_n} x_2^{n-2-l_{n-1}} \cdots \text{ in } \mathfrak{S}_{uw_0}). \end{aligned}$$

Here, $n-k-l_{n+1-k} = n-k-\#\{j < n+1-k : w(j) > w(n+1-k)\} = \#\{j < n+1-k : w(j) < w(n+1-k)\} = \#\{j > k : ww_0(j) < ww_0(k)\} = \mathrm{code}(ww_0)_k$, and thus the number n_{wu} is equal to the coefficient of $x^{\mathrm{code}(ww_0)}$ in $\mathfrak{S}_{uw_0}$. Thus n_{wu} is nonzero only if $(ww_0)^{-1} \underset{\mathrm{lex}}{\geq} (uw_0)^{-1}$, which is equivalent to $w^{-1} \underset{\mathrm{lex}}{\leq} u^{-1}$. Moreover, if $u = w$ then we see that $n_{ww} = 1$. Thus the subquotients of (any) KP filtration of T_w are the modules $\mathcal{S}_u$ ($u^{-1} \underset{\mathrm{lex}}{>} w^{-1}$), together with $\mathcal{S}_w$ which occurs only once. Since $\mathrm{Ext}^1(\mathcal{S}_w, \mathcal{S}_u) = 0$ for $u^{-1} \underset{\mathrm{lex}}{>} w^{-1}$, we can take the filtration to satisfy the additional condition that $\mathcal{S}_w$ occurs as a submodule of T_w. This completes the proof of Lemma 3.1.4. □

Theorem 3.1.1 gives a proof to the classical fact that the product $\mathfrak{S}_\lambda \mathfrak{S}_\mu$ of Schubert polynomials is always a positive sum of Schubert polynomials, whose only previously known proof is through the geometric interpretation of Schubert polynomials. Also we get some corollaries from Theorem 3.1.1:

Corollary 3.1.5. *For any partition λ and an element $\mu \in \mathbb{Z}^n$, the Schur functor image $s_\lambda(\mathcal{S}_\mu)$ of a KP module $\mathcal{S}_\mu$ has a KP filtration. Thus in particular, if we write $\mathfrak{S}_\mu = x^\alpha + x^\beta + \cdots$ as a sum of monoimals then $s_\lambda[\mathfrak{S}_\mu] := s_\lambda(x^\alpha, x^\beta, \ldots)$ (here s_λ stands for a Schur function) is always a positive sum of Schubert polynomials $\mathfrak{S}_\nu$ ($\nu \in \mathbb{Z}^n$).*

Proof. From Theorem 3.1.1 we see that $(\mathcal{S}_\mu)^{\otimes m}$ has a KP filtration for any $m \in \mathbb{Z}_{\geq 0}$. Thus the first claim follows from Corollary 2.3.6(1) since $(\mathcal{S}_\mu)^{\otimes m} \cong \bigoplus_{\lambda \vdash m} s_\lambda(\mathcal{S}_\mu)^{\oplus f^\lambda}$ for certain integers $f^\lambda \geq 1$. The second claim follows since the character of $s_\lambda(\mathcal{S}_\mu)$ is $s_\lambda[\mathfrak{S}_\mu]$. □

Corollary 3.1.6. *In the expansion $\mathfrak{S}_\lambda \mathfrak{S}_\mu = \sum_\nu c^\nu_{\lambda\mu} \mathfrak{S}_\nu$ ($c^\nu_{\lambda\mu} \in \mathbb{Z}$), the coefficient $c^\nu_{\lambda\mu}$ is equal to the dimension of* $\mathrm{Hom}_{\mathfrak{b}}(\mathcal{S}_\lambda \otimes \mathcal{S}_\mu, \mathcal{S}^*_{\rho-\nu} \otimes K_\rho) \cong \mathrm{Hom}_{\mathfrak{b}}(\mathcal{S}_\lambda \otimes \mathcal{S}_\mu \otimes \mathcal{S}_{\rho-\nu}, K_\rho)$. *Similarly, the coefficient $d^\nu_{\lambda\mu} \in \mathbb{Z}$ appearing in the expansion $s_\lambda[\mathfrak{S}_\mu] = \sum_\nu d^\nu_{\lambda\mu} \mathfrak{S}_\nu$ is equal to the dimension of* $\mathrm{Hom}_{\mathfrak{b}}(s_\lambda(\mathcal{S}_\mu), \mathcal{S}^*_{\rho-\nu} \otimes K_\rho) \cong \mathrm{Hom}_{\mathfrak{b}}(s_\lambda(\mathcal{S}_\mu) \otimes \mathcal{S}_{\rho-\nu}, K_\rho)$.

Proof. This is clear from Corollary 2.3.5. ☐

3.2 Explicit filtration for the cases of the Pieri and dual Pieri rules

In this subsection we give explicit forms for KP filtrations of tensor product modules $\mathcal{S}_w \otimes S^d(K^i)$ and $\mathcal{S}_w \otimes \bigwedge^d(K^i)$ ($d \geq 1, 1 \leq i \leq n$). The construction here does not use the results developed in the previous subsection, so it actually gives a proof for Proposition 3.1.2.

First we present Pieri and dual Pieri rules for Schubert polynomials which give expansions of $\mathfrak{S}_w \cdot h_d(x_1, \dots, x_i)$ and $\mathfrak{S}_w \cdot e_d(x_1, \dots, x_i)$ into sums of Schubert polynomials.

Definition 3.2.1. For $w \in S_\infty$, $i \geq 1$ and $d \geq 0$, let

$$X_{i,d}(w) = \{t_{p_1q_1} t_{p_2q_2} \cdots t_{p_dq_d} : p_j \leq i, q_j > i, w_1 \lessdot w_2 \lessdot \cdots, w_1(p_1) < w_2(p_2) < \cdots\}$$

and

$$Y_{i,d}(w) = \{t_{p_1q_1} t_{p_2q_2} \cdots t_{p_dq_d} : p_j \leq i, q_j > i, w_1 \lessdot w_2 \lessdot \cdots, w_1(q_1) > w_2(q_2) > \cdots\}$$

where $w_1 = w, w_2 = wt_{p_1q_1}, w_3 = wt_{p_1q_1}t_{p_2q_2} \cdots$.

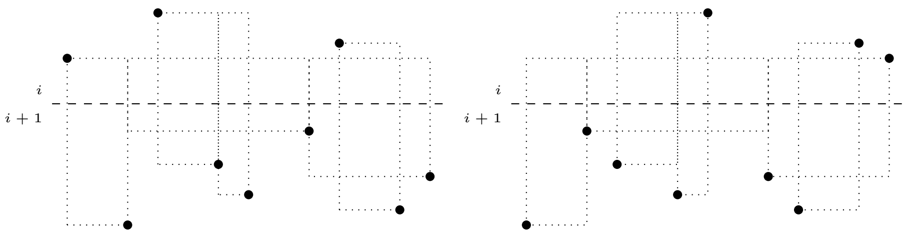


Figure 3: a typical situation for the graphs of w and $w\zeta$ ($\zeta \in X_{i,d}(w)$). Dotted rectangles mean that there are no points of the graphs inside the rectangles. The points of the graphs not shown in the figure are on the same positions.

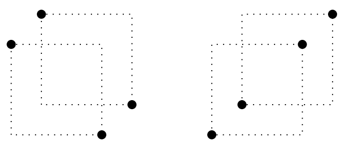


Figure 4: a typical non-situation for w and $w\zeta$ ($\zeta \in X_{i,d}(w)$).

Note that the condition for $X_{i,d}(w)$ (resp. $Y_{i,d}(w)$) implies that $q_1, \dots, q_d$ (resp. $p_1, \dots, p_d$) are all different.

Proposition 3.2.2 (conjectured in [2] and proved in [25])**.** *We have*

$$\mathfrak{S}_w \cdot h_d(x_1, \dots, x_i) = \sum_{\zeta \in X_{i,d}(w)} \mathfrak{S}_{w\zeta}$$

and

$$\mathfrak{S}_w \cdot e_d(x_1, \ldots, x_i) = \sum_{\zeta \in Y_{i,d}(w)} \mathfrak{S}_{w\zeta}.$$

where h_d and e_d denote the complete and elementary symmetric functions respectively. [2] □

Note here that a permutation $\zeta \in X_{i,d}(w)$ (or $Y_{i,d}(w)$) in fact uniquely determines its decomposition into transpositions satisfying the conditions in Definition 3.2.1. So we can write, without ambiguity, for example "for $\zeta = t_{p_1q_1} \cdots t_{p_dq_d} \in X_{i,d}(w)$ define (something) as (some formula involving p_j and q_j)". Hereafter if we write such we will always assume the conditions in Definition 3.2.1.

Now we are going to give explicit forms for KP filtrations of modules $\mathcal{S}_w \otimes S^d(K^i)$ and $\mathcal{S}_w \otimes \bigwedge^d(K^i)$ ($w \in S_\infty^{(n)}$, $d \geq 0$, $1 \leq i \leq n$). Hereafter in this subsection we identify $\mathcal{S}_w$ ($w \in S_\infty^{(n)}$) with a submodule of $T = \bigwedge^\bullet(\bigoplus_{1 \leq i \leq n, j \geq 1} K u_{ij})$ on which $\mathfrak{b}$ acts by $e_{pq}u_{ij} = \delta_{qi}e_{pj}$, by identifying the generator u_w of $\mathcal{S}_w$ with $\bigwedge_{(i,j)\in I(w)} u_{ij}$ (recall that $w \in S_\infty^{(n)}$ implies $I(w) \subset \{1, \ldots, n\} \times \mathbb{Z}_{>0}$). It is easy to see from the definition of KP modules that the submodule of T generated by this element is indeed isomorphic to $\mathcal{S}_w$.

For $1 \leq p \leq q$ we define an operator e'_{qp} acting on T as $e'_{qp}(u_{a_1b_1} \wedge u_{a_2b_2} \wedge \cdots) = \sum_k (\cdots \wedge \delta_{pb_k} u_{a_kq} \wedge \cdots)$. Let these operators act on $T \otimes S^d(K^i)$ and $T \otimes \bigwedge^d(K^i)$ by applying them on the left-hand side tensor component. Also for $j \geq 1$ define an operator $\mu_j : T \otimes \bigotimes^a(K^i) \to T \otimes \bigotimes^{a-1}(K^i)$ ($a \geq 1$) by $u \otimes (v_1 \otimes v_2 \otimes \cdots) \mapsto (\iota_j(v_1) \wedge u) \otimes (v_2 \otimes v_3 \otimes \cdots)$ where $\iota_j(u_p) = u_{pj}$ ($1 \leq p \leq i$). We denote the restrictions of μ_j to $T \otimes S^a(K^i)$ and $T \otimes \bigwedge^a(K^i)$ (seen as submodules of $T \otimes \bigotimes^a(K^i)$) by the same symbol. Note that e'_{rs} and μ_j give $\mathfrak{b}$-endomorphisms on $T \otimes S^\bullet(K^i)$ and $T \otimes \bigwedge^\bullet(K^i)$.

For a permutation z and $p < q$ let $m_{pq}(z) = \#\{r > q : z(p) < z(r) < z(q)\}$ as before, and also define $m'_{qp}(z) = \#\{r < p : z(p) < z(r) < z(q)\}$. For $\zeta = t_{p_1q_1} \cdots t_{p_dq_d} \in X_{i,d}(w)$ (resp. $Y_{i,d}(w)$) define

$$\begin{aligned} v_\zeta &= (\prod_j e_{p_jq_j}^{m_{p_jq_j}(w_j)} u_w) \otimes \prod_j u_{p_j} \\ &= (\prod_j e_{p_jq_j}^{m_{p_jq_j}(w_j)} u_w) \otimes \left(\sum_{\sigma \in S_d} u_{p_{\sigma(1)}} \otimes \cdots \otimes u_{p_{\sigma(d)}} \right) \in \mathcal{S}_w \otimes S^d(K^i) \end{aligned}$$

(resp.

$$\begin{aligned} v_\zeta &= (\prod_j e_{p_jq_j}^{m_{p_jq_j}(w_j)} u_w) \otimes \bigwedge_j u_{p_j} \\ &= (\prod_j e_{p_jq_j}^{m_{p_jq_j}(w_j)} u_w) \otimes \left(\sum_{\sigma \in S_d} \operatorname{sgn}(\sigma) \cdot u_{p_{\sigma(1)}} \otimes \cdots \otimes u_{p_{\sigma(d)}} \right) \in \mathcal{S}_w \otimes \bigwedge^d(K^i) \end{aligned}$$

)

where $w_j = wt_{p_1q_1} \cdots t_{p_{j-1}q_{j-1}}$ as in Definition 3.2.1. Note that these are also

[2]The formulation of dual Pieri rule here is slightly different from the one in [2], but they can be easily seen to be equivalent.

well-defined even if some q_j are greater than n, since in such a case $m_{p_jq_j}(w_j) = 0$. Note also that the products of the operators $e_{p_jq_j}$ above are well-defined since the operators $e_{p_jq_j}$ commute with each others. Also, for such ζ, define a $\mathfrak{b}$-homomorphism $\varphi_\zeta : T \otimes \bigotimes^d(K^i) \to T$ by

$$\varphi_\zeta = \mu_{q_d} \cdots \mu_{q_1} \cdot \prod_j (e'_{q_jp_j})^{m'_{q_jp_j}(w_j)}.$$

Note that the order in the product symbol does not matter since the operators $e'_{q_jp_j}$ commute.

Proposition 3.2.3. *For $\zeta, \zeta' \in X_{i,d}(w)$ (resp. $Y_{i,d}(w)$),*

- *$\varphi_\zeta(v_\zeta)$ is a nonzero multiple of $u_{w\zeta} \in T$, and*
- *$\varphi_{\zeta'}(v_\zeta) = 0$ if $(w\zeta)^{-1} \underset{\text{lex}}{<} (w\zeta')^{-1}$ (resp. $(w\zeta)^{-1} \underset{\text{rlex}}{<} (w\zeta')^{-1}$).*

Let us first see that the constructions for the desired filtrations follow from Proposition 3.2.3.

For a $\mathfrak{b}$-module M and elements $x, y, \ldots, z \in M$ let $\langle x, y, \ldots, z\rangle$ denote the submodule generated by these elements. Consider the sequence of submodules

$$0 \subset \langle v_{\zeta_1}\rangle \subset \langle v_{\zeta_1}, v_{\zeta_2}\rangle \subset \cdots \subset \langle v_\zeta : \zeta \in X_{i,d}(w) \text{ (resp. } Y_{i,d}(w))\rangle$$

inside $\mathcal{S}_w \otimes S^d(K^i)$ (resp. $\mathcal{S}_w \otimes \bigwedge^d(K^i)$), where $\zeta_1, \zeta_2, \ldots \in X_{i,d}(w)$ (resp. $Y_{i,d}(w)$) are all the elements ordered increasingly by the lexicographic (resp. reverse lexicographic) ordering of $(w\zeta)^{-1}$. From the proposition we see that there are surjections $\langle v_{\zeta_1}, \cdots, v_{\zeta_j}\rangle/\langle v_{\zeta_1}, \cdots, v_{\zeta_{j-1}}\rangle \twoheadrightarrow \mathcal{S}_{w\zeta_j}$ induced from φ_{ζ_j}. Thus we have

$$\dim(\mathcal{S}_w \otimes S^d(K^i)) \geq \dim\langle v_\zeta : \zeta \in X_{i,d}(w)\rangle \geq \sum_{\zeta \in X_{i,d}(w)} \dim \mathcal{S}_{w\zeta} = \dim(\mathcal{S}_w \otimes S^d(K^i))$$

and

$$\dim(\mathcal{S}_w \otimes \bigwedge^d(K^i)) \geq \dim\langle v_\zeta : \zeta \in Y_{i,d}(w)\rangle \geq \sum_{\zeta \in Y_{i,d}(w)} \dim \mathcal{S}_{w\zeta} = \dim(\mathcal{S}_w \otimes \bigwedge^d(K^i))$$

respectively, where the last equalities are by Proposition 3.2.2. So the equality must hold everywhere. Thus $\langle v_\zeta : \zeta \in X_{i,d}(w) \text{ (resp. } Y_{i,d}(w))\rangle = \mathcal{S}_w \otimes S^d(K^i)$ (resp. $\mathcal{S}_w \otimes \bigwedge^d(K^i)$) and the surjections above are in fact isomorphisms. So we get:

Theorem 3.2.4.

$$0 \subset \langle v_{\zeta_1}\rangle \subset \langle v_{\zeta_1}, v_{\zeta_2}\rangle \subset \cdots \subset \langle v_\zeta : \zeta \in X_{i,d}(w) \text{ (resp. } Y_{i,d}(w))\rangle$$

gives a KP filtration of $\mathcal{S}_w \otimes S^d(K^i)$ (resp. $\mathcal{S}_w \otimes \bigwedge^d(K^i)$). Explicit isomorphisms $\langle v_{\zeta_1}, \cdots, v_{\zeta_j}\rangle/\langle v_{\zeta_1}, \cdots, v_{\zeta_{j-1}}\rangle \cong \mathcal{S}_{w\zeta_j}$ are given by φ_{ζ_j} defined above. $\square$

Remark 3.2.5. It can be shown that the projective cover of the one dimensional $\mathfrak{b}$-module K_λ ($\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{Z}^n_{\geq 0}$) in the category $\mathcal{C}_{\mathbb{Z}^n_{\geq 0}}$ is given by $S^{\lambda_1}(K^1) \otimes \cdots \otimes S^{\lambda_n}(K^n)$. Thus the construction above for a filtration of $\mathcal{S}_w \otimes S^d(K^i)$ in fact gives a proof to the fact that the projective modules in $\mathcal{C}_{\mathbb{Z}^n_{\geq 0}}$ have KP filtrations, which leads to a different proof from the one in Section 2.3 for the third axiom of highest weight categories (we do not need the results about highest weight structure for $\mathfrak{b}$-modules in the proof of Proposition 3.2.3).

To give a proof for Proposition 3.2.3 we need some lemmas.

For $w \in S^{(n)}_\infty$, $m_{pq}(w) = \#\{r > q : w(p) < w(r) < w(q)\}$ is precisely the number of $r \geq 1$ such that $(q, r) \in I(w)$ and $(p, r) \notin I(w)$. So in particular, if (q, r) in $I(w)$ then $e_{pq}^{m_{pq}(w)} u_w = (\text{const.}) \cdot (u_{pr} \wedge \cdots)$ (it does not matter whether $(p, r) \in I(w)$ or not) and thus $u_{pr} \wedge e_{pq}^{m_{pq}(w)} u_w = 0$. Similarly, if $(r, p) \in I(w)$ then $u_{rq} \wedge (e'_{qp})^{m'_{qp}(w)} u_w = 0$.

Lemma 3.2.6. *Let $w \in S^{(n)}_\infty$ and $i \geq 1$. For $p, p' \leq i$ and $q, q' > i$ such that $\ell(wt_{pq}) = \ell(wt_{p'q'}) = \ell(w)+1$ (i.e. $t_{pq}, t_{p'q'} \in X_{i,1}(w)$), if $u_{pq'} \wedge e_{pq}^{m_{pq}(w)} (e'_{q'p'})^{m'_{q'p'}(w)} u_w \neq 0$ then $w(p') \geq w(p)$ and $w(q') \geq w(q)$, and if $(p, q) = (p', q')$ it is a nonzero multiple of $u_{wt_{pq}}$.* $\square$

Proof. First note that the operations e_{pq}, $e'_{q'p'}$ and $u_{pq'} \wedge -$ on T all commute with all the others. We have the following observations:

(1) If $p < p'$ and $w(p) > w(p')$ then $(p, p') \in I(w)$. Thus in this case $u_{pq'} \wedge e_{pq}^{m_{pq}(w)} (e'_{q'p'})^{m'_{q'p'}(w)} u_w = 0$ since $u_{pq'} \wedge (e'_{q'p'})^{m'_{q'p'}(w)} u_w = 0$.

(2) If $q < q'$ and $w(q) > w(q')$ then $(q, q') \in I(w)$. Thus in this case $u_{pq'} \wedge e_{pq}^{m_{pq}(w)} (e'_{q'p'})^{m'_{q'p'}(w)} u_w = 0$ since $u_{pq'} \wedge e_{pq}^{m_{pq}(w)} u_w = 0$.

(3) If $p < q'$ and $w(p) > w(q')$ then $(p, q') \in I(w)$. Thus in this case $u_{pq'} \wedge e_{pq}^{m_{pq}(w)} (e'_{q'p'})^{m'_{q'p'}(w)} u_w = 0$ since $u_{pq'} \wedge u_w = 0$.

Assume $u_{pq'} \wedge e_{pq}^{m_{pq}(w)} (e'_{q'p'})^{m'_{q'p'}(w)} u_w \neq 0$. First we see that $w(p) < w(q')$ by (3) above. If $w(p') < w(p) < w(q')$ then by $\ell(wt_{p'q'}) = \ell(w) + 1$ we have $p < p'$, but then it contradicts to (1) above. Thus $w(p) \leq w(p')$. By a similar argument (using (2) instead of (1)) we see $w(q) \leq w(q')$. This shows the first claim.

It can be seen that $I(w)$ has the form $I(w) = (\{q\} \times X) \sqcup (Y \times \{p\}) \sqcup I'$ for certain I' where $X = \{r : q < r, w(p) < w(r) < w(q)\}$ and $Y = \{r : r < p, w(p) < w(r) < w(q)\}$ and that $I(wt_{pq}) = (\{p\} \times X) \sqcup (Y \times \{q\}) \sqcup \{(p, q)\} \sqcup I'$. This shows the second claim. $\square$

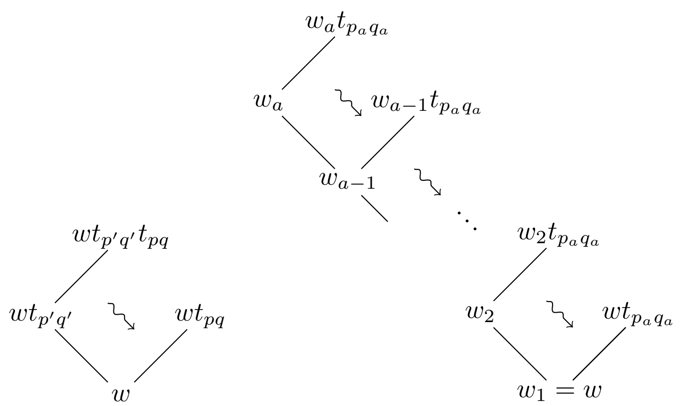
Lemma 3.2.7. *Let $w \in S^{(n)}_\infty$, $i \geq 1$ and $d \geq 0$. Let $\zeta = t_{p_1 q_1} \cdots t_{p_d q_d} \in X_{i,d}(w)$ (resp. $Y_{i,d}(w)$) and $1 \leq a \leq d$. Suppose that there exists no $b < a$ satisfying $p_b = p_a$ (resp. $q_b = q_a$). Then $wt_{p_a q_a} \gtrdot w$, $m_{p_a q_a}(w_a) = m_{p_a q_a}(w)$ and $m'_{q_a p_a}(w_a) = m'_{q_a p_a}(w)$ where $w_a = wt_{p_1 q_1} \cdots t_{p_{a-1} q_{a-1}}$ as in Definition 3.2.1.*

Proof. We show the case $\zeta \in X_{i,d}(w)$: the other case can be treated similarly. First note that $p_1, \dots, p_{a-1} \neq p_a$ by the hypothesis. Also, as we have remarked before, $q_1, \dots, q_a$ are all different. Thus the proof is now reduced to the following lemma. $\square$

Lemma 3.2.8. *Let $p < q$, $p' < q'$ and suppose*

- $\{p, q\} \cap \{p', q'\} = \varnothing$, *and*
- $wt_{p'q'}t_{pq} \gtrdot wt_{p'q'} \gtrdot w$.

Then $m_{pq}(wt_{p'q'}) = m_{pq}(w)$, $m'_{qp}(wt_{p'q'}) = m'_{qp}(w)$ and $wt_{pq} \gtrdot w$.



Proof. Let us begin with a simple observation: suppose there exist two rectangles R_1 and R_2 with edges parallel to coordinate axes. Suppose that no two edges of these rectangles lie on the same line. Then, checking all the possibilities we see that

$$\begin{aligned}&\#(\text{NW and SE corners of } R_1 \text{ lying inside } R_2) - \#(\text{NE and SW corners of } R_1 \text{ lying inside } R_2)\\ &= \#(\text{NW and SE corners of } R_2 \text{ lying inside } R_1) - \#(\text{NE and SW corners of } R_2 \text{ lying inside } R_1).\end{aligned}$$

First consider the case $R_1 = [p, q] \times [w(p), w(q)]$ and $R_2 = [p', q'] \times [w(p'), w(q')]$ in the observation above. $wt_{p'q'}t_{pq} \gtrdot wt_{p'q'} \gtrdot w$ implies that the first term in the left-hand side and the second term in the right-hand side vanish (here the coordinate system is taken so that points with smaller coordinates go NW). Thus all the terms must vanish. In particular the first term on the right-hand side vanishes and thus $wt_{pq} \gtrdot w$.

We have shown that none of the points $(p, w(p)), (p, w(q)), (q, w(p))$ and $(q, w(q))$ lie in $[p', q'] \times [w(p'), w(q')]$, and thus applying the observation to $R_1 = [q, M] \times [w(p), w(q)]$ (resp. $R_1 = [-M, p] \times [w(p), w(q)]$) with $M \gg 0$ and $R_2 = [p', q'] \times [w(p'), w(q')]$ shows $m_{pq}(wt_{p'q'}) = m_{pq}(w)$ (resp. $m'_{qp}(wt_{p'q'}) = m'_{qp}(w)$) since the graphs of w and $wt_{p'q'}$ differ only at the vertices of R_2. □

Proof of Proposition 3.2.3.

Proof for $X_{i,d}(w)$: We assume $(w\zeta)^{-1} \underset{\text{lex}}{\leq} (w\zeta')^{-1}$ and show that $\varphi_{\zeta'}(v_\zeta) = 0$ unless $\zeta' = \zeta$ and $\varphi_\zeta(v_\zeta)$ is a nonzero multiple of $u_{w\zeta}$. Let $\zeta = t_{p_1q_1} \cdots t_{p_dq_d}$ and $\zeta' = t_{p'_1q'_1} \cdots t_{p'_dq'_d}$ as in Definition 3.2.1. We write $w_a = wt_{p_1q_1} \cdots t_{p_{a-1}q_{a-1}}$ and $w'_a = wt_{p'_1q'_1} \cdots t_{p'_{a-1}q'_{a-1}}$.

For $\zeta = \prod_j t_{p_jq_j}$ and $\zeta' = \prod_j t_{p'_jq'_j}$ in $X_{i,d}(w)$ we have

$$\varphi_{\zeta'}(v_\zeta) = \sum_{\sigma \in S_d} \left(u_{p_{\sigma(d)}q'_d} \wedge \cdots \wedge u_{p_{\sigma(1)}q'_1} \wedge (\prod_{j=1}^{d} E_j \prod_{j=1}^{d} E'_j \cdot u_w) \right) \quad \cdots (*)$$

where $E_j = e_{p_j q_j}^{m_{p_j q_j}(w_j)}$ and $E'_j = (e'_{q'_j p'_j})^{m'_{q'_j p'_j}(w'_j)}$.

If $w(p_1) < w(p'_1)$, then $(w\zeta)^{-1}(w(p_1)) = q_1 > p_1 = (w\zeta')^{-1}(w(p_1))$ and $(w\zeta)^{-1}(j) = w^{-1}(j) = (w\zeta')^{-1}(j)$ for all $j < w(p_1)$, and this contradicts the assumption $(w\zeta)^{-1} \underset{\text{lex}}{\leq} (w\zeta')^{-1}$. Thus $w(p_1) \geq w(p'_1)$. Also, by a similar argument, if $p_1 = p'_1$ then $q_1 \leq q'_1$.

Fix $\sigma \in S_d$. Let $1 \leq a \leq d$ be minimal such that $p_a = p_{\sigma(1)}$. Note that this in particular implies $w_a(p_a) = w(p_a)$. We have

$$\begin{aligned}
&u_{p_{\sigma(d)} q'_d} \wedge \cdots \wedge u_{p_{\sigma(1)} q'_1} \wedge (\prod_j E_j \prod_j E'_j \cdot u_w) \\
&= u_{p_{\sigma(d)} q'_d} \wedge \cdots \wedge u_{p_{\sigma(2)} q'_2} \wedge \prod_{j \neq a} E_j \prod_{j \neq 1} E'_j \cdot (u_{p_{\sigma(1)} q'_1} \wedge E_a E'_1 u_w) \\
&= u_{p_{\sigma(d)} q'_d} \wedge \cdots \wedge u_{p_{\sigma(2)} q'_2} \wedge \prod_{j \neq a} E_j \prod_{j \neq 1} E'_j \cdot (u_{p_a q'_1} \wedge E_a E'_1 u_w) \\
&= u_{p_{\sigma(d)} q'_d} \wedge \cdots \wedge u_{p_{\sigma(2)} q'_2} \wedge \prod_{j \neq a} E_j \prod_{j \neq 1} E'_j \cdot (u_{p_a q'_1} \wedge e_{p_a q_a}^{m_{p_a q_a}(w)} (e'_{q'_1 p'_1})^{m'_{q'_1 p'_1}(w)} u_w)
\end{aligned}$$

where the last equality is by Lemma 3.2.7 (note that $w'_1 = w$ by definition).

First consider the case $w(p_1) > w(p'_1)$. We show that the summand in $(*)$ vanishes for all σ. It suffices to show $u_{p_a q'_1} \wedge e_{p_a q_a}^{m_{p_a q_a}(w)} (e'_{q'_1 p'_1})^{m'_{q'_1 p'_1}(w)} u_w = 0$. We have $w(p_a) = w_a(p_a) \geq w(p_1) > w(p'_1)$. Thus by Lemma 3.2.6 we see $u_{p_a q'_1} \wedge e_{p_a q_a}^{m_{p_a q_a}(w)} (e'_{q'_1 p'_1})^{m'_{q'_1 p'_1}(w)} u_w = 0$ (note that $wt_{p_a q_a} \gtrdot w$ by Lemma 3.2.7).

Next consider the case $w(p_1) = w(p'_1)$ and $a > 1$. In this case we see $u_{p_a q'_1} \wedge e_{p_a q_a}^{m_{p_a q_a}(w)} (e'_{q'_1 p'_1})^{m'_{q'_1 p'_1}(w)} u_w = 0$ since $w(p_a) = w_a(p_a) > w(p_1) = w(p'_1)$.

Next consider the case $w(p_1) = w(p'_1)$, $a = 1$ and $q_1 < q'_1$. Then since $wt_{p_1 q_1}, wt_{p'_1 q'_1} \gtrdot w$ it follows that $w(q'_1) < w(q_1)$. So again by Lemma 3.2.6 we see $u_{p_a q'_1} \wedge e_{p_a q_a}^{m_{p_a q_a}(w)} (e'_{q'_1 p'_1})^{m'_{q'_1 p'_1}(w)} u_w = 0$.

So the only remaining summands in $(*)$ are the ones with $(p_1, q_1) = (p'_1, q'_1)$ and $a = 1$, i.e. $p_{\sigma(1)} = p_1$. It is easy to see that the sum of such summands is a nonzero multiple of the sum of terms with $\sigma(1) = 1$. If $\sigma(1) = 1$ we have, by the latter part of Lemma 3.2.6,

$$\begin{aligned}
&u_{p_{\sigma(d)} q'_d} \wedge \cdots \wedge u_{p_{\sigma(1)} q'_1} \wedge (\prod_{j=1}^{d} E_j \prod_{j=1}^{d} E'_j \cdot u_w) \\
&= u_{p_{\sigma(d)} q'_d} \wedge \cdots \wedge u_{p_{\sigma(2)} q'_2} \wedge \prod_{j=2}^{d} E_j \prod_{j=2}^{d} E'_j \cdot (u_{p_1 q_1} \wedge e_{p_1 q_1}^{m_{p_1 q_1}(w)} (e'_{q_1 p_1})^{m'_{q_1 p_1}(w)} u_w) \\
&= (\text{nonzero const.}) \cdot u_{p_{\sigma(d)} q'_d} \wedge \cdots \wedge u_{p_{\sigma(2)} q'_2} \wedge (\prod_{j=2}^{d} E_j \prod_{j=2}^{d} E'_j \cdot u_{wt_{p_1 q_1}}).
\end{aligned}$$

So, working inductively on d (using $wt_{p_1 q_1}$, $t_{p_2 q_2} \cdots t_{p_d q_d}$ and $t_{p'_2 q'_2} \cdots t_{p'_d q'_d}$ in place of w, ζ and ζ' respectively, noting that if $(p_1, q_1) = (p'_1, q'_1)$ then $(w\zeta)^{-1} \underset{\text{lex}}{\leq} (w\zeta')^{-1}$ implies $((wt_{p_1 q_1}) \cdot t_{p_2 q_2} \cdots t_{p_d q_d})^{-1} = (w\zeta)^{-1} \underset{\text{lex}}{\leq} (w\zeta')^{-1} = ((wt_{p_1 q_1}) \cdot t_{p'_2 q'_2} \cdots t_{p'_d q'_d})^{-1})$ we see that:

- $u_{p_{\sigma(d)}q'_d} \wedge \cdots \wedge u_{p_{\sigma(1)}q'_1} \wedge (\prod_j E_j \prod_j E'_j \cdot u_w)$ vanishes if $(w\zeta)^{-1} \underset{\text{lex}}{<} (w\zeta')^{-1}$, or if $\zeta' = \zeta$ and $\sigma \neq \mathrm{id}$, and
- if $\zeta' = \zeta$ and $\sigma = \mathrm{id}$ then it is a nonzero multiple of $u_{w\zeta}$.

This finishes the proof for $X_{i,d}(w)$.

Proof for $Y_{i,d}(w)$: This proceeds much similarly to the previous case. Here instead of $(*)$ we use

$$\begin{aligned}\varphi_{\zeta'}(v_\zeta) &= \sum_{\sigma \in S_d} \left(\mathrm{sgn}(\sigma) \cdot u_{p_{\sigma(d)}q'_d} \wedge \cdots \wedge u_{p_{\sigma(1)}q'_1} \wedge (\prod_{j=1}^{d} E_j \prod_{j=1}^{d} E'_j \cdot u_w) \right) \\ &= \sum_{\sigma \in S_d} \left(u_{p_d q'_{\sigma^{-1}(d)}} \wedge \cdots \wedge u_{p_1 q'_{\sigma^{-1}(1)}} \wedge (\prod_{j=1}^{d} E_j \prod_{j=1}^{d} E'_j \cdot u_w) \right)\end{aligned}$$

where $E_j = e_{p_j q_j}^{m_{p_j q_j}(w_j)}$ and $E'_j = (e'_{q'_j p'_j})^{m'_{q'_j p'_j}(w'_j)}$ as before.

We assume $(w\zeta)^{-1} \underset{\text{rlex}}{\leq} (w\zeta')^{-1}$. Fix σ and take $1 \leq a \leq d$ minimal with $q'_a = q'_{\sigma^{-1}(1)}$. By a similar argument to the above, it suffices to show that $u_{p_1 q'_a} \wedge e_{p_1 q_1}^{m_{p_1 q_1}(w)} (e'_{q'_a p'_a})^{m'_{q'_a p'_a}(w)} u_w$ is zero unless $a = 1$ and $(p'_1, q'_1) = (p_1, q_1)$, and in a such case it is a nonzero multiple of $u_{wt_{p_1 q_1}}$.

Since $(w\zeta)^{-1} \underset{\text{rlex}}{\leq} (w\zeta')^{-1}$ by the hypothesis, we see that $w(q_1) \geq w(q'_1)$, and if $w(q_1) = w(q'_1)$ then $p_1 \leq p'_1$.

If $w(q_1) > w(q'_1)$ then the claim follows from Lemma 3.2.6 since $w(q_1) > w(q'_1) \geq w'_a(q'_a) = w(q'_a)$. If $w(q_1) = w(q'_1)$ and $a > 1$ then the claim follows from Lemma 3.2.6 since in this case $w(q_1) = w(q'_1) > w(q'_a)$ by $wt_{p_1 q_1}, wt_{p'_1 q'_1} \gtrdot w$. If $q_1 = q'_1$, $a = 1$ and $p_1 < p'_1$ the claim follows from Lemma 3.2.6 since $w(p_1) > w(p'_1)$. Finally if $(p_1, q_1) = (p'_1, q'_1)$ and $a = 1$ then $u_{p_1 q'_a} \wedge e_{p_1 q_1}^{m_{p_1 q_1}(w)} (e'_{q'_a p'_a})^{m'_{q'_a p'_a}(w)} u_w = u_{p_1 q_1} \wedge e_{p_1 q_1}^{m_{p_1 q_1}(w)} (e'_{q_1 p_1})^{m'_{q_1 p_1}(w)} u_w$ is a constant multiple of $u_{wt_{p_1 q_1}}$ by Lemma 3.2.6. □

Remark 3.2.9. Although the Pieri rule for KP modules we have shown above (Theorem 3.2.4) implies Monk's rule for KP modules (Proposition 3.1.2), we actually need a slightly more precise result in the next section. Here we give it as a remark.

For $\zeta = t_{pq} \in X_{i,1}(w) = Y_{i,1}(w)$ $(p \leq i < q)$ we have $v_\zeta = v_{pq} = e_{pq}^{m_{pq}(w)} u_w \otimes u_p \in \mathcal{S}_w \otimes K^i$ and $\varphi_\zeta = \varphi_{pq} = \mu_q \cdot (e'_{qp})^{m'_{qp}(w)}$. In this case we see directly from Lemma 3.2.6 that $\varphi_{p'q'}(v_{pq}) = 0$ unless both $w(p) \leq w(p')$ and $w(q) \leq w(q')$ hold.

Let $(p_1, q_1), \ldots, (p_r, q_r)$ be all the elements in $X_{i,1}(w) = Y_{i,1}(w)$, indexed so that there exist no $a < b$ such that $w(p_a) \leq w(p_b)$ and $w(q_a) \leq w(q_b)$ hold simultaneously. Then by the same argument as before in this subsection, $0 \subset \langle v_1 \rangle \subset \langle v_1, v_2 \rangle \subset \cdots \subset \langle v_1, \ldots, v_r \rangle$ $(v_j = v_{p_j q_j})$ gives a KP filtration of $\mathcal{S}_w \otimes K^i$ with $\langle v_1, \ldots, v_j \rangle / \langle v_1, \ldots v_{j-1} \rangle \cong \mathcal{S}_{wt_{p_j q_j}}$.

4 Kraśkiewicz-Pragacz modules and Ringel duality

In this section we study a special case $\mathcal{C}_n = \mathcal{C}_{\Lambda_n}$ of our highest weight categories. We prove that the Ringel dual of $\mathcal{C}_n$ is equivalent to itself, and that the anti-autoequivalence $\mathcal{C}_n^\Delta \to \mathcal{C}_n^\Delta$ given by this duality preserves a certain natural tensor product operation on $\mathcal{C}_n^\Delta$.

4.1 Ringel dual of the highest weight category $\mathcal{C}_n$

Let $\mathcal{C}_n = \mathcal{C}_{\Lambda_n}$ (recall that $\Lambda_n = \{(a_1, \ldots, a_n) : 0 \leq a_i \leq n-i\}$ is an order ideal in $\mathbb{Z}^n$). Since $\Lambda_n = \{\mathrm{code}(w) : w \in S_n\}$ we see that the standard objects in $\mathcal{C}_n$ are $\mathcal{S}_w$ ($w \in S_n$). In this subsection we show that the highest weight category $\mathcal{C}_n$ is self Ringel-dual. Precisely, we show the following:

Theorem 4.1.1. *The Ringel dual of the highest weight category $\mathcal{C}_n$ is equivalent to $\mathcal{C}_n$ itself. The functor F in Section 1.3.6 acts on the standard modules by $F(\mathcal{S}_w) = \mathcal{S}_{w_0 w w_0}$ ($w \in S_n$).*

From the theorem in particular we obtain the following symmetry relation for the Hom and Ext groups between KP modules:

Corollary 4.1.2. $\mathrm{Ext}^i_{\mathcal{C}_n}(\mathcal{S}_w, \mathcal{S}_v) \cong \mathrm{Ext}^i_{\mathcal{C}_n}(\mathcal{S}_{w_0 v w_0}, \mathcal{S}_{w_0 w w_0})$ *for any $w, v \in S_n$ and any $i \geq 0$.* $\square$

Let us move to the proof of Theorem 4.1.1. First we prepare some definitions and results. For $\lambda = \mathrm{code}(w) \in \Lambda_n$ define $\overline{\lambda} = \mathrm{code}(w_0 w w_0)$. Note that by definition, for $\lambda, \mu \in \Lambda_n$, $\lambda \leq \mu$ iff $\overline{\lambda} \geq' \overline{\mu}$. For each $\lambda \in \Lambda_n$, define $T(\lambda) = \bigotimes_{1 \leq j \leq n-1} \bigwedge^{\overline{\lambda}_j} K^{n-j}$.

As we showed in the proof of Lemma 3.1.4, $T(\lambda)$ has a filtration whose subquotients are standard modules $\mathcal{S}_\mu$ ($\mu \in \Lambda_n, \mu \preceq \lambda$). Since $\overline{\rho - \lambda} = \rho - \overline{\lambda}$ we have $T(\lambda) \cong T(\rho - \lambda)^* \otimes K_\rho$, and thus $T(\lambda)$ also has a filtration whose subquotients are costandard modules $\mathcal{S}^*_{\rho-\nu} \otimes K_\rho$ ($\nu \in \Lambda_n, \nu \preceq \lambda$). Thus by Proposition 1.3.8 we see that $\mathrm{Ext}^1(\mathcal{S}_\mu, T(\lambda)) = 0$ for all $\mu \in \Lambda_n$. Thus $T(\lambda)$ is a tilting in $\mathcal{C}_n$.

Since the weights of $\mathcal{S}_\mu$ are all $\preceq \mu$, the weights of $T(\lambda)$ are all $\preceq \lambda$ and the weight space $T(\lambda)_\lambda$ is one-dimensional. By these properties we see that $T(\lambda)$ contains the indecomposable tilting module corresponding to λ (in fact, we will see that $T(\lambda)$ is an indecomposable tilting). So if we define $T = \bigoplus_{\lambda \in \Lambda_n} T(\lambda) \cong \bigwedge^\bullet(K^{n-1} \oplus K^{n-2} \oplus \cdots \oplus K^1)$, then T is a full tilting (beware that the definition of T here is slightly different from the one in the previous section).

Like in Section 3.2 we make use of the operators e'_{ij} on T. Let $\mathfrak{b}' = \bigoplus_{i \leq j} K e'_{ij}$ be a copy of $\mathfrak{b}$. Take a basis $\{u_{ij} : i, j \geq 1, i+j \leq n\}$ of $K^{n-1} \oplus \cdots \oplus K^1$ so that the action of $\mathfrak{b}$ is given by $e_{pq} u_{ij} = \delta_{qi} u_{pj}$. Then we define the action of $\mathfrak{b}'$ on $K^{n-1} \oplus \cdots \oplus K^1$ by $e'_{pq} u_{ij} = \delta_{qj} u_{ip}$, and define the action on T as the one induced from this action. In other words, if $-' : T \to T$ is the involution given by $u'_{ij} = u_{ji}$, then $e'_{ij} = -' \circ e_{ij} \circ -'$. Note that, like in Section 3.2, the actions of $\mathfrak{b}$ and $\mathfrak{b}'$ commute with each other.

By Proposition 1.3.11, if M has a KP filtration, then $\mathrm{Ker}(M^{\leq' \lambda} \twoheadrightarrow M^{<' \lambda})$ is isomorphic to a direct sum of copies of $\mathcal{S}_\lambda$, where $M^{\leq' \lambda}$ and $M^{<' \lambda}$ are the largest quotients of M whose weights are all $\leq' \lambda$ and $<' \lambda$ respectively. In this case it can be seen that the isomorphism can be written as $\mathcal{S}_\lambda \otimes (M^{\leq'\lambda})_\lambda \ni xu_\lambda \otimes v \mapsto xv \in \mathrm{Ker}(M^{\leq'\lambda} \twoheadrightarrow M^{<'\lambda})$, where on the left-hand side $\mathfrak{b}$ acts only on $\mathcal{S}_\lambda$.

Proof of Theorem 4.1.1. Let $\mathcal{C} = \mathcal{C}_n$. Throughout this proof and thereafter we write Hom, End and Ext^i for $\mathrm{Hom}_\mathcal{C}$, $\mathrm{End}_\mathcal{C}$ and $\mathrm{Ext}^i_\mathcal{C}$ respectively.

Since the actions of $\mathfrak{b}$ and $\mathfrak{b}'$ commute, we have an algebra homomorphism $\mathcal{U}(\mathfrak{b}) \cong \mathcal{U}(\mathfrak{b}') \to \mathrm{End}_\mathfrak{b}(T)$, and thus an $\mathrm{End}_\mathfrak{b}(T)$-module can be naturally seen as a $\mathcal{U}(\mathfrak{b})$-module (note that, as we have remarked before, we will simply write $\mathrm{End}(T)$ to mean $\mathrm{End}_\mathfrak{b}(T) = \mathrm{End}_\mathcal{C}(T)$ hereafter). If M is an $\mathrm{End}(T)$-module, then its weight-space decomposition as a $\mathcal{U}(\mathfrak{b})$-module is given by $M = \bigoplus_{\lambda\in\Lambda_n} M_\lambda = \bigoplus_{\lambda\in\Lambda_n} \pi_\lambda M$, where $\pi_\lambda \in \mathrm{End}(T)$ is the projection $T = \bigoplus_{\mu\in\Lambda_n} T(\mu) \twoheadrightarrow T(\lambda)$; in particular the weights of M are all in Λ_n. So we have a functor $\mathcal{C}^\vee = \mathrm{End}(T)\text{-mod} \to \mathcal{C}$. We want to show that this functor is an equivalence and the composition $\mathcal{C} \overset{F=\mathrm{Hom}(-,T)}{\longrightarrow} \mathcal{C}^\vee \overset{\sim}{\to} \mathcal{C}$ sends $\mathcal{S}_w$ to $\mathcal{S}_{w_0ww_0}$.

First we show the second claim. By definition, $\mathcal{S}_w$ is isomorphic to the $\mathfrak{b}$-submodule of T generated by $\bigwedge_{i<j, w(i)>w(j)} u_{i,n+1-j}$; hereafter we identify u_w with this element. Note that $u'_w = \pm u_{w_0ww_0}$. We have an injective homomorphism $\mathcal{S}_{w_0ww_0} \to \mathrm{Hom}(\mathcal{S}_w, T)$ given by $xu_{w_0ww_0} \mapsto (v \mapsto x'v)$ $(x \in \mathcal{U}(\mathfrak{b}))$: it is well-defined since $xu_{w_0ww_0} = 0$ implies $x'yu_w = \pm y(xu_{w_0ww_0})' = 0$ for any $y \in \mathcal{U}(\mathfrak{b})$, and it is injective because $v \mapsto x'v$ maps $u_w = \pm u'_{w_0ww_0}$ to $\pm(xu_{w_0ww_0})'$. Since T has a costandard filtration, by Proposition 1.3.10 the dimension of $\mathrm{Hom}(\mathcal{S}_w, T)$ is equal to the number of times the costandard module $\mathcal{S}^*_{w_0w} \otimes K_\rho$ appears in (any) costandard filtration of T. Since $T \cong T^* \otimes K_\rho$, this number is equal to the number of times $\mathcal{S}_{w_0w}$ appears in (any) standard filtration of T. From Cauchy identity we see that $\mathrm{ch}(T) = \prod (x_i+1)^{n-i} = \sum_{v\in S_n} \mathfrak{S}_v(x)\mathfrak{S}_{vw_0}(1)$, and thus we see that $\dim \mathrm{Hom}(\mathcal{S}_w, T) = \mathfrak{S}_{w_0ww_0}(1) = \dim \mathcal{S}_{w_0ww_0}$. So the injection above is in fact an isomorphism and this shows the second claim.

Now let us show that the functor $\mathcal{C}^\vee \to \mathcal{C}$ given above is an equivalence. First we note the following thing. Define an algebra $A = \mathcal{U}(\mathfrak{b})/I$, where I is the two-sided ideal generated by all elements in $\mathcal{U}(\mathfrak{h}) = S(\mathfrak{h}) \cong K[\mathfrak{h}^*]$ which vanish on Λ_n (here $\Lambda_n \subset \mathbb{Z}^n$ is identified with a subset of $\mathfrak{h}^*$ via the pairing $\langle \lambda, h\rangle = \sum_i \lambda_i h_i$ introduced before). Then the objects in $\mathcal{C}$, i.e. weight $\mathfrak{b}$-modules with weights in Λ_n, are just the finite dimensional A-modules (note that A-modules automatically have weight decompositions since any element $p_\lambda \in K[\mathfrak{h}^*]$ such that $p_\lambda(\mu) = \delta_{\lambda\mu}$ $(\forall \mu \in \Lambda_n)$ acts as a projection onto the λ-weight space). Thus it suffices to show that the map

$$\varphi : A \ni a \mapsto (\mathfrak{b}'\text{-action of } a \text{ on } T) \in \mathrm{End}(T)$$

is an isomorphism. We note here that A has an algebra anti-automorphism ι defined by $\iota(h) = \langle \rho, h\rangle - h$ $(h \in \mathfrak{h})$ and $\iota(e_{ij}) = -e_{ij}$ $(1 \leq i < j \leq n)$. For each $\lambda \in \Lambda_n$ take $p_\lambda \in A$ as above. Note that $\iota(p_\lambda) = p_{\rho-\lambda}$.

Let $0 \leq d \leq \binom{n}{2}$. It suffices to show that φ induces an isomorphism between $A_d := \sum_{\lambda_1+\cdots+\lambda_n=d} Ap_\lambda A$ and $\mathrm{End}(T)_d := \mathrm{End}(\bigwedge^d(K^{n-1} \oplus \cdots \oplus K^1))$, since as algebras $A = \bigoplus_d A_d$ (this follows easily from $hp_\lambda = p_\lambda h$ and $e_{ij}p_\lambda = p_{\lambda-\alpha_{ij}}e_{ij}$) and $\mathrm{End}(T) = \bigoplus_d \mathrm{End}(T)_d$. So let us fix such d hereafter in this proof. Let the elements of $\{\lambda \in \Lambda_n : \sum \lambda_i = d\}$ be $\lambda^{(1)} > \lambda^{(2)} > \cdots > \lambda^{(r)}$. Note

$\overline{\lambda^{(1)}} <' \overline{\lambda^{(2)}} <' \cdots <' \overline{\lambda^{(r)}}$. Define $I_k = \sum_{\mu \geq \lambda^{(k)}} Ap_\mu A$. Also define $J_k = \mathrm{Hom}(T^{\leq' \overline{\lambda^{(k)}}}, T) \subset \mathrm{End}(T)$ where $T^{\leq' \overline{\lambda^{(k)}}}$ is the largest quotient of T whose weights are all $\leq' \overline{\lambda^{(k)}}$. In other words, J_k consists of all morphisms in $\mathrm{End}(T)$ which vanishes on the weight spaces T_μ ($\mu \not\leq' \overline{\lambda^{(k)}}$). Define $I_0 = 0$ and $J_0 = 0$. Note that $I_0 \subset \cdots \subset I_r = A_d$ and $J_0 \subset \cdots \subset J_r = \mathrm{End}(T)_d$. It suffices to show that $\varphi(I_k) \subset J_k$ and that φ induces an isomorphism $I_k/I_{k-1} \to J_k/J_{k-1}$ for all $1 \leq k \leq r$.

Fix $1 \leq k \leq r$. Let $\lambda = \lambda^{(k)}$. The first claim $\varphi(I_k) \subset J_k$ follows since for $\mu \geq \lambda$, p_μ acts on T as the projection onto $T(\overline{\mu})$, and every weight ν of $T(\overline{\mu})$ satisfies $\nu \leq' \overline{\mu} \leq' \overline{\lambda}$. Let us now show that the induced map $I_k/I_{k-1} \to J_k/J_{k-1}$ is an isomorphism. We show that I_k/I_{k-1} and J_k/J_{k-1} are both isomorphic to $\mathcal{S}_\lambda \otimes \mathcal{S}_{\rho-\lambda}$ as vector spaces and that the composition of isomorphisms $I_k/I_{k-1} \cong \mathcal{S}_\lambda \otimes \mathcal{S}_{\rho-\lambda} \cong J_k/J_{k-1}$ coincides with the map induced from φ.

We first show that $I_k/I_{k-1} \cong \mathcal{S}_\lambda \otimes \mathcal{S}_{\rho-\lambda}$. First note that A is a projective object in $\mathcal{C} = A$-mod. Since projective objects in $\mathcal{C}$ have standard filtrations, A has a standard filtration.

By definition, $A/I_k \cong A^{<\lambda}$ and $A/I_{k-1} \cong A^{\leq\lambda}$, and thus $I_k/I_{k-1} \cong \mathrm{Ker}(A^{\leq\lambda} \to A^{<\lambda})$. By Proposition 1.3.11 this is a direct sum of m copies of $\mathcal{S}_\lambda$, where m is the number of times $\mathcal{S}_\lambda$ appears in a standard filtration of A. This number m can be calculated, by Proposition 1.3.10, as $\dim \mathrm{Hom}(A, \mathcal{S}^*_{\rho-\lambda} \otimes K_\rho) = \dim(\mathcal{S}^*_{\rho-\lambda} \otimes K_\rho) = \dim \mathcal{S}_{\rho-\lambda}$. Thus $\mathcal{S}_\lambda \otimes \mathcal{S}_{\rho-\lambda}$ and I_k/I_{k-1} have the same dimensions. We claim that the map $\mathcal{S}_\lambda \otimes \mathcal{S}_{\rho-\lambda} \ni xu_\lambda \otimes yu_{\rho-\lambda} \mapsto xp_\lambda\iota(y) = \iota(y\iota(xp_\lambda)) \in I_k/I_{k-1}$ is well-defined. To see this, first observe that the weights of I_k/I_{k-1} (resp. $\iota(I_k/I_{k-1})$ are all $\leq \lambda$ (resp. $\leq' \rho - \lambda$). Thus the submodule of I_k/I_{k-1} (resp. $\iota(I_k/I_{k-1})$) generated by $p_\lambda\iota(y)$ (resp. $\iota(xp_\lambda)$) is a quotient of $\mathcal{S}_\lambda$ (resp. $\mathcal{S}_{\rho-\lambda}$) by Remark 2.2.5, and thus $xu_\lambda = 0$ (resp. $yu_{\rho-\lambda} = 0$) implies $xp_\lambda\iota(y) = 0$ (resp. $y\iota(xp_\lambda) = 0$). This verifies the well-definedness of the map above. It is clear that the map above is a surjection. By the equality of dimensions this is in fact an isomorphism.

Next we show $J_k/J_{k-1} \cong \mathcal{S}_\lambda \otimes \mathcal{S}_{\rho-\lambda}$. Since $T^{<'\overline{\lambda}}$ has a standard filtration by Proposition 1.3.11, $\mathrm{Ext}^1(T^{<'\overline{\lambda}}, T)$ vanishes. So $J_k/J_{k-1} \cong \mathrm{Hom}(\mathrm{Ker}(T^{\leq'\overline{\lambda}} \twoheadrightarrow T^{<'\overline{\lambda}}), T)$ via the restriction map. The right-hand side is isomorphic to $\mathrm{Hom}(\mathcal{S}_{\overline{\lambda}} \otimes (T^{\leq'\overline{\lambda}})_{\overline{\lambda}}, T) \cong ((T^{\leq'\overline{\lambda}})_{\overline{\lambda}})^* \otimes \mathrm{Hom}(\mathcal{S}_{\overline{\lambda}}, T)$ by the remark before the proof. As we have seen above, $\mathrm{Hom}(\mathcal{S}_{\overline{\lambda}}, T) \cong \mathcal{S}_\lambda$. On the other hand, since $T \cong T^* \otimes K_\rho$, $((T^{\leq'\overline{\lambda}})_{\overline{\lambda}})^* \cong (T_{\leq\mu})_\mu$ where $\mu = \rho - \overline{\lambda}$ and $T_{\leq\mu}$ denotes the largest submodule of T whose weights are $\leq \mu$. Since $\mathcal{S}_\mu \cong (P_\mu)^{\leq\mu}$ we have $(T_{\leq\mu})_\mu \cong \mathrm{Hom}(P_\mu, T_{\leq\mu}) \cong \mathrm{Hom}(\mathcal{S}_\mu, T_{\leq\mu}) \cong \mathrm{Hom}(\mathcal{S}_\mu, T) \cong \mathcal{S}_{\overline{\mu}} = \mathcal{S}_{\rho-\lambda}$.

Now we show that the composition of these isomorphisms coincides with the map induced from φ, up to a sign depending only on λ. Chasing the isomorphisms we see that it suffices to show $\varphi(xp_\lambda y)(\tau) = \langle \iota(y)' u_{\rho-\overline{\lambda}}, \tau\rangle x' u_{\overline{\lambda}}$, up to a sign depending only on λ, for all $\tau \in T_{\overline{\lambda}}$ and all $x, y \in A$, where $\langle -, - \rangle$ is a natural bilinear form on T defined by $T \otimes T \overset{\text{mult.}}{\to} T = \bigwedge^\bullet(K^{n-1} \oplus \cdots \oplus K^1) \twoheadrightarrow \bigwedge^{\binom{n}{2}}(K^{n-1} \oplus \cdots \oplus K^1) \cong K$. Note that from the definition we see that $\langle u, x'v \rangle = \langle \iota(x)'u, v \rangle$ holds for any $u, v \in T$ and $x \in A$. First we have $\varphi(xp_\lambda y)(\tau) = x'p'_\lambda y'\tau$. Here $p'_\lambda y'\tau \in T(\overline{\lambda})_{\overline{\lambda}}$ so it must be a constant multiple of $u_{\overline{\lambda}}$. Using the pairing defined above we see that this is equal to $\pm\langle p'_\lambda y'\tau, u_{\rho-\overline{\lambda}}\rangle u_{\overline{\lambda}}$ with the sign depending only on λ, since $u_{\overline{\lambda}} \wedge u_{\rho-\overline{\lambda}} = \pm u_{w_0}$.

Since $\langle p'_\lambda y'\tau, u_{\rho-\overline{\lambda}}\rangle = \langle \tau, \iota(y)' p'_{\rho-\lambda} u_{\rho-\overline{\lambda}}\rangle = \langle \tau, \iota(y)' u_{\rho-\overline{\lambda}}\rangle$ we are done. □

4.2 Compatibility with tensor product

In this subsection we show that the Ringel duality functor $F = \mathrm{Hom}(-, T)$ and the tensor product operation on $\mathcal{C}^\Delta$ in some sense commute with each other. Precisely, we show the following:

Theorem 4.2.1. *Let $M, N \in \mathcal{C}_n$ have standard filtrations. Then $F((M \otimes N)^{\Lambda_n}) \cong (FM \otimes FN)^{\Lambda_n}$, where for a weight $\mathfrak{b}$-module L, $L^{\Lambda_n} \in \mathcal{C}_n$ denotes the largest quotient of L which is in $\mathcal{C}_n$.*

Let $\mathcal{C}_+$ be the category of all finite dimensional weight $\mathfrak{b}$-modules whose weights are in $\mathbb{Z}^n_{\geq 0}$. Note that if $M, N \in \mathcal{C}_+$ then $M \otimes N \in \mathcal{C}_+$. Using the terminology from highest weight categories we say that $M \in \mathcal{C}_+$ has a standard filtration if M has a filtration whose successive quotients are of the form $\mathcal{S}_\lambda$ ($\lambda \in \mathbb{Z}^n_{\geq 0}$). Note that, as we showed in Section 3.1, if $M, N \in \mathcal{C}_+$ have standard filtrations then $M \otimes N$ also has a standard filtration.

Remark 4.2.2. If $L \in \mathcal{C}_+$ has a standard filtration, then as we show below, $\mathrm{ch}(L^{\Lambda_n}) = \mathrm{ch}(L)$ holds in the ring H_n. So, together with Theorem 4.1.1, this theorem can be seen as a module theoretic counterpart of Proposition 1.1.4; i.e. the claim that $\mathfrak{S}_w \mapsto \mathfrak{S}_{w_0 w w_0}$ is a ring automorphism on H_n.

First we prepare some lemmas.

Lemma 4.2.3. *Let $\iota : H_n \to H_n$ be the ring automorphism in Proposition 1.1.4. If $M \in \mathcal{C}_+$ has a standard filtration, then $\mathrm{ch}(FM) = \iota(\mathrm{ch}(M))$ in H_n.*

Proof. Since the extensions of KP modules with T vanish, if we have an exact sequence $0 \to L \to M \to N \to 0$ with $L, M, N \in \mathcal{C}_+$ having standard filtrations, then $0 \to FN \to FM \to FL \to 0$ is exact. Thus we only have to show the lemma for $M = \mathcal{S}_w$ ($w \in S^{(n)}_\infty$).

The case $w \in S_n$ follows from Theorem 4.1.1. If $w \in S^{(n)}_\infty \setminus S_n$ then we have $F\mathcal{S}_w = \mathrm{Hom}(\mathcal{S}_w, T) = 0$ since $\mathcal{S}_w$ is generated by an element of weight $\mathrm{code}(w)$ while the weight space $T_{\mathrm{code}(w)}$ is zero. Thus the lemma follows for this case since $\mathfrak{S}_w = 0$ in H_n. □

Lemma 4.2.4. *Let $M \in \mathcal{C}_+$ have a standard filtration. Then $\mathrm{ch}(M^{\Lambda_n}) = \mathrm{ch}(M)$ as elements of H_n. If $\overline{M} \in \mathcal{C}_n$ is a quotient of M and $\mathrm{ch}(\overline{M}) = \mathrm{ch}(M)$ in H_n, then $\overline{M} \cong M^{\Lambda_n}$.*

Proof. By Proposition 1.3.11, $\mathrm{Ker}(M \twoheadrightarrow M^{\Lambda_n})$ has a filtration whose subquotients are of the form $\mathcal{S}_v$ ($v \in S^{(n)}_\infty \setminus S_n$). Thus $\mathrm{ch}(M) = \mathrm{ch}(M^{\Lambda_n}) +$ (a linear combination of $\mathfrak{S}_v$ ($v \in S^{(n)}_\infty \setminus S_n$)), and the second term vanishes in H_n by Proposition 1.1.6. The second claim follows from the first claim since $\bigoplus_{\lambda \in \Lambda_n} \mathbb{Z}x^\lambda \cong H_n$. □

Lemma 4.2.5. *Let $M, N \in \mathcal{C}_+$ have standard filtrations. Suppose that the morphism $FM \otimes FN \to F(M \otimes N)$ given by $\varphi \otimes \psi \mapsto (m \otimes n \mapsto \varphi(m) \wedge \psi(n))$ is surjective. Then it induces an isomorphism $(FM \otimes FN)^{\Lambda_n} \cong F(M \otimes N)$ ($\cong F((M \otimes N)^{\Lambda_n})$).*

Proof. We have, as vector spaces, $F(M\otimes N) = \mathrm{Hom}(M\otimes N, T) = \bigoplus_{\lambda\in\Lambda_n} \mathrm{Hom}(M\otimes N, T(\lambda))$. It can be seen that $\mathrm{Hom}(M\otimes N, T(\lambda))$ is the $\overline{\lambda}$-weight space of the $\mathfrak{b}$-module $F(M\otimes N)$. Thus $F(M\otimes N)\in\mathcal{C}_n$. By Lemma 4.2.3 we have, in H_n, $\mathrm{ch}(F(M\otimes N)) = \iota(\mathrm{ch}(M)\mathrm{ch}(N)) = \iota(\mathrm{ch}(M))\iota(\mathrm{ch}(N)) = \mathrm{ch}(FM\otimes FN)$. Thus the claim follows from the second statement in Lemma 4.2.4. □

For $M, N \in \mathcal{C}_+$ having standard filtrations, let $\mathcal{P}(M,N)$ be the claim that the map $FM\otimes FN \to F(M\otimes N)$ above is surjective (and thus $(FM\otimes FN)^{\Lambda_n} \cong F((M\otimes N)^{\Lambda_n})$).

Lemma 4.2.6. *Let $L, M, N, X \in \mathcal{C}_+$ have standard filtrations. Then the following implications hold:*

(1) If L is a direct sum component of M then $\mathcal{P}(M,X)$ implies $\mathcal{P}(L,X)$.

(2) Suppose that there exists an exact sequence $0\to L\to M\to N\to 0$. Then $\mathcal{P}(L,X)\wedge\mathcal{P}(N,X) \implies \mathcal{P}(M,X)$ and $\mathcal{P}(M,X) \implies \mathcal{P}(L,X)$ hold (in fact $\mathcal{P}(M,X)$ also implies $\mathcal{P}(N,X)$, but we do not need it here).

(3) $\mathcal{P}(L,M)$ and $\mathcal{P}(L\otimes M, N)$ implies $\mathcal{P}(L, M\otimes N)$.

Proof. (1) is clear since F preserves direct sums.

(2) We have a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & FN\otimes FX & \longrightarrow & FM\otimes FX & \longrightarrow & FL\otimes FX & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & F(N\otimes X) & \longrightarrow & F(M\otimes X) & \longrightarrow & F(L\otimes X) & \longrightarrow & 0. \end{array}$$

Here the rows are exact since $\mathrm{Ext}^1(N,T)$ and $\mathrm{Ext}^1(N\otimes X, T)$ vanish. This shows $\mathcal{P}(L,X)\wedge\mathcal{P}(N,X) \implies \mathcal{P}(M,X)$ and $\mathcal{P}(M,X) \implies \mathcal{P}(L,X)$.

(3) This holds since

$$\begin{array}{ccc} FL\otimes FM\otimes FN & \longrightarrow & F(L\otimes M)\otimes FN \\ \downarrow & & \downarrow \\ FL\otimes F(M\otimes N) & \longrightarrow & F(L\otimes M\otimes N) \end{array}$$

commutes.

□

Lemma 4.2.7. *Let $M\in\mathcal{C}_+$ have a standard filtration. Let $\lambda\in\Lambda_n$. Let $V\subset \mathrm{Hom}(M,T)$ be the submodule consisting of all homomorphisms which vanish on the μ-weight spaces for any $\mu>\lambda$ (it is a submodule since the action of $\mathfrak{b}'$ on T preserves weights with respect to $\mathfrak{h}\subset\mathfrak{b}$). Then $\mathrm{Hom}(M,T)/V \cong \mathrm{Hom}(M,T)^{<'\overline{\lambda}}$, the largest quotient of $\mathrm{Hom}(M,T)$ whose weights are all $<'\overline{\lambda}$ (recall that for $\lambda = \mathrm{code}(w)\in\Lambda_n$ we defined $\overline{\lambda} = \mathrm{code}(w_0ww_0)$).*

Proof. It suffices to show that the characters of both sides coincide.

First note that $V = \mathrm{Hom}(M^{\not> \lambda}, T)$ where $M^{\not>\lambda}$ is the largest quotient of M whose weights are all $\not> \lambda$. From Proposition 1.3.11 we see that $M^{\not>\lambda}$ has a standard filtration and, if $\mathrm{ch}(M) = \sum_\mu c_\mu \mathfrak{S}_\mu$, then the number of times $\mathcal{S}_\mu$ appears in a standard filtration of $M^{\not>\lambda}$ is c_μ if $\mu \not> \lambda$ and 0 otherwise. Thus we see from Theorem 4.1.1 that $\mathrm{ch}(V) = \mathrm{ch}(\mathrm{Hom}(M^{\not>\lambda}, T)) = \sum_{\mu\in\Lambda_n, \mu\not>\lambda} c_\mu \mathfrak{S}_{\overline{\mu}}$. We also see from Theorem 4.1.1 that $\mathrm{Hom}(M, T)$ has a a standard filtration with $\mathcal{S}_\mu$ appearing $c_{\overline{\mu}}$ times for each $\mu \in \Lambda_n$. Thus $\mathrm{ch}(\mathrm{Hom}(M, T)) = \sum_{\mu\in\Lambda_n} c_\mu \mathfrak{S}_{\overline{\mu}}$. So $\mathrm{ch}(\mathrm{Hom}(M, T)/V) = \sum_{\mu\in\Lambda_n, \mu>\lambda} c_\mu \mathfrak{S}_{\overline{\mu}}$.

On the other hand, since $\mathrm{Hom}(M, T)$ has a standard filtration, by Proposition 1.3.11 we see $\mathrm{ch}(\mathrm{Hom}(M, T)^{<'\overline{\lambda}}) = \sum_{\mu\in\Lambda_n, \mu<'\overline{\lambda}} c_{\overline{\mu}} \mathfrak{S}_\mu = \sum_{\mu\in\Lambda_n, \overline{\mu}<'\overline{\lambda}} c_\mu \mathfrak{S}_{\overline{\mu}} = \sum_{\mu\in\Lambda_n, \mu>\lambda} c_\mu \mathfrak{S}_{\overline{\mu}} = \mathrm{ch}(\mathrm{Hom}(M, T)/V)$. This shows the claim. $\square$

Recall from the proof of Theorem 4.1.1 that T has an action of $\mathfrak{b}'$, a copy of $\mathfrak{b}$, defined by $e'_{ij} u_{pq} = \delta_{jq} u_{pi}$, which commutes with the usual action of $\mathfrak{b}$. Recall also that we have identified u_w with $\bigwedge_{(i,j)\in J(w)} u_{i,j} \in T$ where $J(w) = \{(i, \overline{j}) : i < j, w(i) > w(j)\}$.

We write $\overline{w} = w_0 w w_0$ $(w \in S_n)$ and $\overline{k} = n + 1 - k$ $(1 \le k \le n)$.

Lemma 4.2.8. *Let $w \in S_n$ and $1 \le i \le n - 1$. For $1 \le p, p' \le i$ and $i + 1 \le q, q' \le n$ such that $\ell(wt_{pq}) = \ell(wt_{p'q'}) = \ell(w) + 1$, if $(e'_{\overline{q'},\overline{p'}})^{m_{\overline{q'},\overline{p'}}(\overline{w})} e_{pq}^{m_{pq}(w)} u_w \wedge u_{p,\overline{q'}} \neq 0$ then $w(p) \le w(p')$ and $w(q) \le w(q')$. Moreover, $(e'_{\overline{q},\overline{p}})^{m_{\overline{q},\overline{p}}(\overline{w})} e_{pq}^{m_{pq}(w)} u_w \wedge u_{p,\overline{q}}$ is a nonzero multiple of $u_{wt_{pq}}$.*

Proof. This is essentially the same as Lemma 3.2.6. $\square$

Proof of Theorem 4.2.1. First we show that $\mathcal{P}(\mathcal{S}_w, \mathcal{S}_{s_i})$ holds for any $w \in S_n$ and any $1 \le i \le n - 1$.

Recall that the isomorphism $\mathcal{S}_{\overline{w}} \to \mathrm{Hom}(\mathcal{S}_w, T)$ was given by $xu_{\overline{w}} \mapsto (v \mapsto x'v)$. Thus we want to show that the map $\varphi : \mathcal{S}_{\overline{w}} \otimes K^{n-i} \to F(\mathcal{S}_w \otimes K^i)$ given by $yu_{\overline{w}} \otimes u_q \mapsto (xu_w \otimes u_p \mapsto xy'u_w \wedge u_{pq})$ is a surjection.

Let $(p_1, q_1), \dots, (p_r, q_r)$ be all the pairs (p, q) such that $1 \le p \le i < q \le n$ and $\ell(wt_{pq}) = \ell(w) + 1$, ordered by the lexicographic order of $(w(p), w(q))$. Let $w^k = wt_{p_k q_k}$. Then $\mathrm{code}(w^1) < \cdots < \mathrm{code}(w^r)$ and $\mathrm{code}(\overline{w^1}) >' \cdots >' \mathrm{code}(\overline{w^r})$.

For an $x \in S_n$ and $1 \le p < q \le n+1$ such that $\ell(xt_{pq}) = \ell(x) + 1$, let $v_{pq}(x) = e_{pq}^{m_{pq}(x)} u_x \otimes u_p \in \mathcal{S}_x \otimes K^n$ (note that this definition is also valid for $q = n+1$ since $m_{pq}(x) = 0$ in such case). Note that $v_{pq}(x)$ has weight $\mathrm{code}(xt_{pq})$. Note that by Remark 3.2.9, $\{v_{pq}(x) : 1 \le p \le i < q \le n + 1, \ell(xt_{pq}) = \ell(x) + 1\}$ generates $\mathcal{S}_x \otimes K^i$ as a $\mathfrak{b}$-module.

For $0 \le k \le r$, let U_k be the submodule of $\mathcal{S}_w \otimes K^i$ generated by $v_{p_l,q_l}(w)$ $(l > k)$ together with $v_{j,n+1}(w)$ $(1 \le j \le i, \ell(wt_{j,n+1}) = \ell(w) + 1)$. Note that $U_0 = \mathcal{S}_w \otimes K^i$. From Remark 3.2.9 we see that $U_{k-1}/U_k \cong \mathcal{S}_{w^k}$. In particular the weights of U_{k-1}/U_k is all $\le \mathrm{code}(w^k)$, and since $\mathrm{code}(w^1) \le \cdots \le \mathrm{code}(w^k)$ we see that the weights of $(\mathcal{S}_w \otimes K^i)/U_k$ are all $\le \mathrm{code}(w^k)$. Moreover, U_r has a filtration by modules $\mathcal{S}_{wt_{j,n+1}}$, and thus $\mathrm{ch}(U_r) = 0$ in H_n. Therefore $(\mathcal{S}_w \otimes K^i)/U_r \cong (\mathcal{S}_w \otimes K^i)^{\Lambda_n}$ by Lemma 4.2.4 (note that $(\mathcal{S}_w \otimes K^i)/U_r \in \mathcal{C}_n$ since $\mathcal{S}_{w^1}, \dots, \mathcal{S}_{w^r} \in \mathcal{C}_n$).

Let V_k $(k = 1, \dots, r)$ be the submodule of $F(\mathcal{S}_w \otimes K^i) = \mathrm{Hom}(\mathcal{S}_w \otimes K^i, T)$ consisting of the homomorphisms which vanish on the μ-weight spaces for any

$\mu >$ code(w^k). By Lemma 4.2.7, $F(\mathcal{S}_w \otimes K^i)/V_k \cong F(\mathcal{S}_w \otimes K^i)^{<' \mathrm{code}(\overline{w^k})}$ $(1 \le k \le r)$. We see $V_r = F(\mathcal{S}_w \otimes K^i)$ since by the argument above the weights of $(\mathcal{S}_w \otimes K^i)^{\Lambda_n}$ are all $\le$ code(w^r). We also set $V_0 = 0$.

Note that the constituents in a standard filtration of $F(\mathcal{S}_w \otimes K^i)$ are $\mathcal{S}_{\overline{w^1}}, \dots, \mathcal{S}_{\overline{w^r}}$ by Theorem 4.1.1. In particular, the only constituent $\mathcal{S}_x$ with code$(\overline{w^{k-1}}) >'$ code$(x) \ge'$ code$(\overline{w^k})$ is $\mathcal{S}_{\overline{w^k}}$. Thus $V_k/V_{k-1} \cong \mathrm{Ker}(F(\mathcal{S}_w \otimes K^i)^{<' \mathrm{code}(\overline{w^{k-1}})} \twoheadrightarrow F(\mathcal{S}_w \otimes K^i)^{<' \mathrm{code}(\overline{w^k})}) \cong \mathcal{S}_{\overline{w^k}}$ by Proposition 1.3.11. In particular any nonzero element of weight code$(\overline{w^k})$ in V_k/V_{k-1} generates V_k/V_{k-1}.

We show $\varphi(v_{\overline{q_k},\overline{p_k}}(\overline{w})) \in V_k \smallsetminus V_{k-1}$ for each k. Note that the desired surjectivity of φ follows from this claim since it shows that $\varphi(v_{\overline{q_k},\overline{p_k}}(\overline{w})) + V_{k-1}$ is a cyclic generator of V_k/V_{k-1}, i.e. $\mathcal{U}(\mathfrak{b})(\varphi(v_{\overline{q_k},\overline{p_k}}(\overline{w})) + V_{k-1}) = V_k$.

For $1 \le k, l \le r$ we have $\varphi(v_{\overline{q_k},\overline{p_k}}(\overline{w}))(v_{p_l,q_l}(w)) = (e_{p_l q_l}^{m_{p_l q_l}(w)} (e'_{\overline{q_k},\overline{p_k}})^{m_{\overline{q_k},\overline{p_k}}(\overline{w})} u_w) \wedge u_{p_l,\overline{q_k}}$. By Lemma 4.2.8, if $\varphi(v_{\overline{q_k},\overline{p_k}}(\overline{w}))(v_{p_l,q_l}(w)) \neq 0$ then $w(p_l) \le w(p_k)$ and $w(q_l) \le w(q_k)$ and thus in particular $l \le k$. Thus $\varphi(v_{\overline{q_k},\overline{p_k}}(\overline{w}))$ induces a map $(\mathcal{S}_w \otimes K^i)/U_k \to T$ (note that the elements $v_{j,n+1}(w)$ obviously vanish under $\varphi(v_{\overline{q_k},\overline{p_k}}(\overline{w}))$ since T does not have the corresponding weights). Since the weights of $(\mathcal{S}_w \otimes K^i)/U_k$ are all $\le$ code(w^k), this shows $\varphi(v_{\overline{q_k},\overline{p_k}}(\overline{w})) \in V_k$. Moreover $\varphi(v_{\overline{q_k},\overline{p_k}}(\overline{w}))(v_{p_k,q_k}(w)) \neq 0$ by Lemma 4.2.8, and since $v_{p_k,q_k}(w)$ has weight code$(\overline{w^k})$ this shows $\varphi(v_{\overline{q_k},\overline{p_k}}(\overline{w})) \notin V_{k-1}$. Therefore we checked the claim and thus $\mathcal{P}(\mathcal{S}_w, \mathcal{S}_{s_i})$ follows.

Now we can proceed to the general case. From (2) of Lemma 4.2.6 we see that $\mathcal{P}(M, \mathcal{S}_{s_i})$ holds for any M having a standard filtration. Since if M has a standard filtration then $M \otimes \mathcal{S}_{s_i}$ also has a standard filtration, (3) of Lemma 4.2.6 shows that $\mathcal{P}(M, \mathcal{S}_{s_i} \otimes \mathcal{S}_{s_j} \otimes \cdots)$ holds for any $i, j, \dots$ and any M. Then from (1) of Lemma 4.2.6 we see that $\mathcal{P}(M, T(\lambda))$ holds for any λ and any M, since $T(\lambda)$ is a direct sum component of $\bigotimes_{1 \le i \le n-1} (\mathcal{S}_{s_{n-i}})^{\otimes \overline{\lambda}_i}$. Thus again from (2) of Lemma 4.2.6 we get $\mathcal{P}(M, \mathcal{S}_\lambda)$, since as we showed in Lemma 3.1.4 there is an injection $\mathcal{S}_\lambda \hookrightarrow T(\lambda)$ such that its cokernel admits a standard filtration. Thus $\mathcal{P}(M, N)$ for general M, N follows by (2) of Lemma 4.2.6. $\square$

Remark 4.2.9. As we saw, $(M, N) \mapsto (M \otimes N)^{\Lambda_n}$ is a very fundamental operation in the category $\mathcal{C}_n^{\Delta}$; this in fact defines a structure of symmetric monoidal category on $\mathcal{C}_n^{\Delta}$. Experimental results suggest an interesting conjecture relating this "restricted tensor product" operation and our full tilting module T: the dimension of $(T^{\otimes k})^{\Lambda_n}$ seems to be $(k+1)^{\binom{n}{2}}$ for any k. Also there is a finer form of this conjecture: the dimension of the degree-d piece (with respect to the grading induced from the natural grading on $T = \bigwedge^{\bullet}(\cdots)$) of $(T^{\otimes k})^{\Lambda_n}$ seems to be $k^d \binom{\binom{n}{2}}{d}$. Note that these conjectures can actually be rephrased to a combinatorial conjecture on Schubert polynomials by Lemma 4.2.4.

It can be shown that the latter version of the conjecture implies that $\mathrm{Hom}(T^{\otimes k}, T)$ also has a dimension $(k+1)^{\binom{n}{2}}$. Note that this is true for $k = 1$ since $\mathrm{ch}(\mathrm{End}(T)) = \iota(\mathrm{ch}(T)) = \sum_{v \in S_n} \mathfrak{S}_v(x) \mathfrak{S}_{w_0 v}(1)$ and thus $\dim(\mathrm{End}(T)) = \sum_{v \in S_n} \mathfrak{S}_v(1) \mathfrak{S}_{w_0 v}(1) = \sum_{v \in S_n} \mathfrak{S}_{v^{-1}}(x) \mathfrak{S}_{(w_0 v)^{-1}}(1) = \sum_{v \in S_n} \mathfrak{S}_v(1) \mathfrak{S}_{v w_0}(1) = 2^{\binom{n}{2}}$ by Cauchy formula.

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